1 Choi and Shin (1992)

Consider a model with two firms and two vertically differentiated goods. We label h and l each firm. The h (resp. l) firm produces the high (resp. low) quality variant u_h (resp. u_l). The range of quality is in the interval $[\bar{u}, 0]$ where $\bar{u}$ is the highest quality level that is technologically feasible.There is a continuum of consumers distributed uniformly over the interval $[\theta-1,\theta\ ]$ with unit density, where $\theta > 1$. Each consumer does one of three things -buy from firm 1, buy from firm 2, or not buy at all. We consider a 3-stage game with the high quality firm defining its variant at the first stage, then the low quality firm defining u_L and finally the two firms deciding simultaneously at the third stage the corresponding prices p_h and p_l. Consumers display the same preferences with respect to variants, so that their indirect utility function $U_i(\theta)$ writes as

$$U_i(\theta) = \begin{cases} \theta u_h - p_h & \text{if she buys h} \\ \theta u_l - p_l & \text{if she buys l} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The corresponding demand function of each firm can be written as follows

$$x_h(p_h, p_l) = (\theta - \frac{p_h - p_l}{u_h - u_l})$$
$$x_l(p_h, p_l) = \left(\frac{p_h - p_l}{u_h - u_l} - \frac{p_l}{u_l}\right).$$

Profit functions then write as

$$\Pi_h = p_h x_h$$
$$\Pi_l = p_l x_l$$

From profit maximization w.r.t p_i

$$\frac{\partial}{\partial p_h}\left(p_h(\theta - \frac{p_h - p_l}{u_h - u_l})\right) = \frac{(p_l - 2p_h + \theta u_h - \theta u_l)}{u_h - u_l} = 0$$

From the F.O.C. we get:

$$\frac{(p_l - 2p_h + \theta u_h - \theta u_l)}{u_h - u_l} = 0 \text{ or}$$
$$p_h(p_l) = \frac{1}{2}p_l + \frac{1}{2}\theta u_h - \frac{1}{2}\theta u_l$$

Notice that $p_h(p_l)$ is a BR, namely best reply function.

Can you comment this BR?

Now I compute the BR for firm l :

$$\frac{\partial}{\partial p_l}\left(p_l\left(\frac{p_h-p_l}{u_h-u_l}-\frac{p_l}{u_l}\right)\right)=\frac{1}{u_l\left(u_h-u_l\right)}\left(p_h u_l-2p_l u_h\right)=0$$

$$\begin{aligned}\frac{1}{u_l\left(u_h-u_l\right)}\left(p_h u_l-2p_l u_h\right)=0 \text{ or}&\\ p_l(p_h)=\frac{1}{2}\frac{p_h}{u_h}u_l&\end{aligned}$$

Again, how do you interprete $p_l(p_h)$? It is a BR (prices are strategic complements!)

$\begin{array}{cc}\frac{1}{2}\frac{p_h}{u_h}u_l-p_l & 0\\ \frac{1}{2}p_l+\frac{1}{2}\theta u_h-\frac{1}{2}\theta u_l-p_h & 0\end{array}$, Solution is:

$$p_h=\frac{(2\theta u_h^2-2\theta u_h u_l)}{4u_h-u_l}$$
$$p_l=\frac{(\theta u_h u_l-\theta u_l^2)}{4u_h-u_l}$$

Thus, we can write the demand function at these optimal price. Let me remind that we are still at the price stage.

$$x_h(p_h,p_l)=(\theta-\frac{p_h-p_l}{u_h-u_l})$$
$$x_l(p_h,p_l)=\left(\frac{p_h-p_l}{u_h-u_l}-\frac{p_l}{u_l}\right)$$

Hence, when evaluating the demand function at these prices we obtain:

$$x_h(p_h,p_l)=\left[(\theta-\frac{p_h-p_l}{u_h-u_l})\right]_{p_h=\frac{\left(2\theta u_h^2-2\theta u_h u_l\right)}{4u_h-u_l},p_l=\frac{\left(\theta u_h u_l-\theta u_l^2\right)}{4u_h-u_l}}=2\theta\frac{u_h}{4u_h-u_l}$$
$$x_l(p_h,p_l)=\left[\left(\frac{p_h-p_l}{u_h-u_l}-\frac{p_l}{u_l}\right)\right]_{p_h=\frac{\left(2\theta u_h^2-2\theta u_h u_l\right)}{4u_h-u_l},p_l=\frac{\left(\theta u_h u_l-\theta u_l^2\right)}{4u_h-u_l}}=\theta\frac{u_h}{4u_h-u_l}.$$

Accordingly, we can write the profit function which will be considered by firms at the quality stage.

$$\Pi_h=\left(2\theta\frac{u_h}{4u_h-u_l}\right)\left(\frac{(2\theta u_h^2-2\theta u_h u_l)}{4u_h-u_l}\right)$$

and

$$\Pi_l = \left(\theta \frac{u_h}{4u_h - u_l}\right)\left(\frac{(\theta u_h u_l - \theta u_l^2)}{4u_h - u_l}\right).$$

In line with the above, I proceed by backward induction.

So I start considering the low quality firm. This firm defines at the second stage, after observing the optimal quality

$$\frac{\partial}{\partial u_l}\left(\left(\theta \frac{u_h}{4u_h - u_l}\right)\left(\frac{(\theta u_h u_l - \theta u_l^2)}{4u_h - u_l}\right)\right) = \theta^2 u_h^2 \frac{4u_h - 7u_l}{(4u_h - u_l)^3} = 0.$$

We get that

$$u_l(u_h) = \frac{4}{7}u_h$$

THIS IS NOT THE OPTIMAL QUALITY! For the optimal quality to be computed, we have to move to the first stage e verify how the high-quality firm determines u_h.

$$\frac{\partial}{\partial u_h}\left(\left(2\theta \frac{u_h}{4u_h - u_l}\right)\left(\frac{(2\theta u_h^2 - 2\theta u_h u_l)}{4u_h - u_l}\right)\right) = 4\theta^2 u_h \frac{4u_h^2 + 2u_l^2 - 3u_h u_l}{(4u_h - u_l)^3}.$$

It is immediate to notice that profit function Π_h is monotonically increasing in u_h. Since we have assumed nil costs, we can conclude that

$$u_h^* = \bar{u}.$$

Then, we move to stage 2 and finally stage 3.

As far as stage 2, we can write

$$u_l^* = \frac{4}{7}\bar{u}.$$

As far as the third stage we can write

$$p_h = \left[\frac{(2\theta u_h^2 - 2\theta u_h u_l)}{4u_h - u_l}\right]_{u_h=\bar{u}, u_l=\frac{4}{7}\bar{u}} = \frac{1}{4}\bar{u}\theta$$

$$p_l = \left[\frac{(\theta u_h u_l - \theta u_l^2)}{4u_h - u_l}\right]_{u_h=\bar{u}, u_l=\frac{4}{7}\bar{u}} = \frac{1}{14}\bar{u}\theta.$$

Now, it remains to check whether the assumption that the market is uncovered is consistent with this equilibrium configuration.

To this aim, we proceed as follows.

Let us writing first the marginal consumer who is indifferent from buying u_l ant not buying at all.

$$\theta_l = \frac{u_l}{p_l}$$

$$\theta_l = \left[\frac{p_l}{u_l}\right]_{p_l=\frac{1}{14}\bar{u}\theta, u_l=\frac{4}{7}\bar{u}} = \frac{\theta}{8}.$$

Hence, market coverage is depending on whether $\frac{\theta}{8} \gtreqless \theta - 1$. $\frac{\theta}{8} - \theta + 1 = -\frac{1}{8}(7\theta - 8)$.

Then

- for any θ such that $(8 - 7\theta) > 0$ or $\theta < \frac{8}{7}$ the market is uncovered
- for any θ such that $(8 - 7\theta) \leq 0$ or $\theta \geqslant \frac{8}{7}$ the market is covered.

2 Bertrand

$$p_h = \left[\left[\left(2\theta u_h^2 - 2\theta u_h u_l\right)\right]_{u_h=u_l} = 0\right]_{u_h=u_l}$$
$$p_l = \left[\left(\theta u_h u_l - \theta u_l^2\right)\right]_{u_h=u_l} = 0$$

When moving to the profits:

$$\Pi_h = \left(2\theta\frac{u_h}{4u_h - u_l}\right)\left(\frac{\left(2\theta u_h^2 - 2\theta u_h u_l\right)}{4u_h - u_l}\right)$$

$\left[\left(2\theta\frac{u_h}{4u_h-u_l}\right)\left(\frac{\left(2\theta u_h^2-2\theta u_h u_l\right)}{4u_h-u_l}\right)\right]_{u_h=u_l} = 0$

and

$$\Pi_l = \left(\theta\frac{u_h}{4u_h - u_l}\right)\left(\frac{\left(\theta u_h u_l - \theta u_l^2\right)}{4u_h - u_l}\right).$$

$\left[\left(\theta\frac{u_h}{4u_h-u_l}\right)\left(\frac{\left(\theta u_h u_l-\theta u_l^2\right)}{4u_h-u_l}\right)\right]_{u_h=u_l} = 0 = 0$

Market power:

Lerner index:

$$L = \frac{p - c}{p}$$

Its value ranges from 0, in case of a perfect competition, to 1, in case of a pure monopoly.

Definition 1 *Bertrand paradox: in a duopoly with homogeneous product and firms competing in price, the equilibrium configuration "coincides" with the one observed under perfect competition.*