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Natural Interest Rate and Asset Price Bubbles: How Bubbles Counteract Low Interest Rates*

Jacopo Bonchi†

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Abstract

By developing a three-period OLG model with rational asset price bubbles and non-neutral monetary policy, I show how bubbles prevent low interest rates, when the natural rate of interest declines permanently. Bubbles push the natural interest rate up by serving as store of value (*saving channel*) and collateral (*borrowing channel*), and this avoids a long-lasting ZLB episode. Bubbles reallocate resources across generations too, and this reallocation implies welfare losses. These results shed light on the pattern of the US risk-free interest rates and on that of net worth and consumption across generations before the Great Recession.

JEL Classification Numbers: E13, E32, E44, E52

Keywords: Asset price bubbles; Natural interest rate; Zero lower bound

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1 Introduction

The natural interest rate is the level of the real interest rate consistent with potential output and stable prices (Wicksell, 1898). Its estimates for the US economy point to a historical decline, which is caused by structural changes in saving and investment, and it is transmitted to risk-free real and nominal interest rates at different maturities (Summers, 2014; IMF, 2014; Laubach and Williams, 2016).[1] Although the declining trend of the natural interest rate started a long before the Great Recession, persistently low interest rates appeared only in the wake of the 2007-2008 crisis.[2] A possible explanation to this fact is that falling interest rates were counteracted by asset price bubbles between the mid-1990s and the mid-2000s. In this paper I investigate the effect of asset price bubbles on interest rates, when the natural rate of interest is declining. The analysis explains the pattern of the US risk-free interest rates before the global financial crisis and that of net worth and real consumption across generations in the same period.

Four stylized facts characterized the US economy in the decade preceding the Great Recession. First, stocks and houses strongly appreciated despite their fundamental value rose only slightly. The large fluctuations in asset prices were driven by a "bubble" component, which was completely unrelated to the fundamentals. Second, there was a shift in the composition of assets portfolio towards risky assets, along with a decrease in the saving rate and a sustained increase in private debt. Households saved less and invested massively their savings in stocks and houses, which were also used as collateral in the credit market, fostering debt accumulation. Third, the declining trend of interest rates slowed down temporarily. Fourth, the pattern of net worth and real consumption differed radically according to the age cohort. Specifically, net worth and consumption increased more in the middle-aged and old cohorts than in the young cohorts.

I provide a theoretical framework to interpret these facts jointly. In particular, this paper shows that the emergence of bubbles can be a possible explanation for the absence of low interest rates between the mid-1990s and the mid-2000s, as well as for the pattern of wealth and consumption across generations in the same period. The intuition is straightforward. On the one hand, the appre-

[1] An alternative view is expressed by Borio (2012) and Lo and Rogoff (2015), who emphasize the role of monetary and financial factors in determining low interest rates.

[2] Laubach and Williams (2016) show a downward trend in the natural rate of interest starting from the 1980s, while Eggertsson et al. (2019) document that the structural forces behind this trend were already at work in the 1970s.

ciation of stocks and houses induced households to save less because of the higher expected lifetime income, and to divert funds away from risk-free assets because of the higher return of risky ones. On the other hand, it raised the value of households' collaterals, stimulating borrowing. As a result of the reallocation and reduction of saving and the increased demand for borrowing, the natural interest rate raised, pushing real and nominal rates up.[3] Furthermore, wealth and consumption gains from asset price bubbles were unevenly distributed across generations due to the life-cycle pattern of assets and debt. As young cohorts have a lower income, they borrow more and own less assets than older cohorts. Hence bubbles caused a large increase in net worth of middle-aged and old cohorts, which spurred their consumption. Young cohorts instead took on more debt via appreciated collaterals than other cohorts, but their wealth and consumption rose less.

In order to explain the effect of bubbles on interest rates, I develop an OLG model with three generations, non-neutral monetary policy and rational asset price bubbles. This theoretical framework is particularly suitable for my analysis, because it allows to introduce easily low interest rates and asset price bubbles, as well as to replicate the life-cycle pattern of saving and net worth. As agents get a positive income only in middle age, middle-aged households supply funds to young ones in exchange for a risk-free bond. The natural interest rate is accordingly determined by credit demand and supply, and it can be permanently negative when there is a structural lack of demand. If the natural interest rate turns negative in a bubbleless economy, price stability prevents the central bank from driving the real interest rate to its natural level, and a long-lasting zero lower bound (ZLB) episode makes risk-free nominal and real interest rates persistently low. However, the supply of bonds is constrained by an exogenous debt limit and the resulting shortage of assets fosters rational bubbles.

There are different varieties of bubbles, which are distinguished through the time period they are introduced. Each period middle-aged households initiate a new variety of bubbly assets and purchase the old varieties from old households. As bubbles transfer resources to old age, they serve as a store of value. Furthermore, young households do not operate in the bubbles market, but they

[3] Government debt and public pension schemes can raise interest rates too. However, the increase in the US public debt over the last forty years was not sufficient to offset the declining trend of interest rates (Eggertsson et al., 2019). This is especially true for the period between the mid-1990s and the mid-2000s, when the US federal debt as a percent of GDP declined slightly according to FRED data. Furthermore, the US public spending for pension did not change radically before and after the financial crisis, but it ranged from 6% to 7% percent of GDP over the period 1990-2013 (OECD data). It is accordingly implausible that the public pensions counteracted the fall in the US natural rate of interest.

can borrow against the value of the bubble they will initiate, given that bubble creation will improve their ability to repay debt in the next period. Therefore the new bubble is a collateral.

Bubbles push the natural interest rate up by absorbing excess saving and fostering borrowing through two channels, which are related to their capacity of being simultaneously store of value and collateral. An additional store of value causes a reallocation of savings from bonds to bubbly assets. It also induces to save less, because it raises the lifetime income by providing further earnings in old age. This is the *saving channel*. On the other hand, bubbly collateral relaxes the borrowing constraint and increases the amount of debt through the *borrowing channel*. Even if aggregate demand is scarce, the upward pressure of bubbles keeps the natural interest rate positive, avoiding low interest rates. Finally, bubbles redistribute resources from young households to old ones. As fewer funds are supplied by middle-aged households in the credit market, young households consume less in a bubbly economy than in a bubbleless one. So the higher demand for borrowing from the young generation translates in higher interest payments and debt, but not in more funds to finance consumption. The old generation in contrast consumes more, because the selling of bubbly assets and the higher return from lending increase its income. Given that the consumption losses during youth outweigh the consumption gains in old age, the reallocation of resources implemented by bubbles reduces the life-cycle utility of the representative agent compared to a bubbleless economy.

My paper relates to two strands of the economic literature. First, it is based on the recent literature on secular stagnation (e.g., Summers 2013, 2014, 2015; Baldwin and Teulings, 2014; Gordon, 2015; Eggertsson et al., 2016; Bacchetta et al., 2016; and Eggertsson et al., 2019). I add rational bubbles to the theoretical model of Eggertsson et al. (2019), to formalize the idea of Summers (2013) that low interest rates were postponed by asset price bubbles. Bacchetta et al. (2016) also find that bubbles can bring the economy out of the ZLB, but they abstract from bubbly collateral which is central in my paper.

Second, my work is inspired by the extended literature on rational bubbles in the OLG setting, which includes, among others, Samuelson (1958), Tirole (1985), Martin and Ventura (2011, 2012), Galì (2014) and Asriyan et al. (2016). Bubbly assets originate in my model from a shortage of investment opportunities, like in the traditional OLG models of rational bubbles (e.g., Samuelson, 1958; and Tirole, 1985). However, the bubble is not welfare enhancing, because it undermines the transfer of resources to young age in a three-period model with borrowing constrained young households. Tirole (1985) analyzes the role of asset price bubbles as store of value in a real economy, while Martin and Ventura (2011) study the effect of bubbly collateral on borrowing in a similar model augmented with financial frictions. I extend their analysis to a monetary economy without capital, in which the natural interest rate is declining and the nominal interest rate is constrained by the ZLB. I model bubble creation and destruction in a way similar to Galì (2014) to investigate how bubbles counteract low interest rates, neglecting the macroeconomic instability associated with bubbly episodes. Asriyan et al. (2016) show that bubbly collateral can prevent a liquidity trap, but they do not study explicitly the mechanisms through which bubbles raise interest rates and their implications for the allocation of resources across generations.

The paper is organized as follows. I present in section 2 empirical evidence regarding the stylized facts. In section 3 I outline the model through which I explain these facts, while I define its equilibrium in section 4. Section 5 concludes.

2 Empirical Evidence

The movements in stock and house prices between the mid-1990s and the mid-2000s were large and mainly driven by a bubble (e.g., Leroy, 2004; Shiller, 2007). The market capitalization of the US listed companies was 90% of GDP in 1995 and it rose up to around 140% in 2007 (World Bank data). House prices followed a similar trend, as witnessed by the Case-Shiller national home price index which more than doubled in the same period.[4] Against this backdrop, saving and borrowing behaviors changed, along with the pattern of interest rates and that of net worth and real consumption across generations.

The personal saving rate declined from 6.3% in 1994 to 3% in 2007, according to FRED data. Equally households reallocated their assets portfolio towards stocks and houses by diverting funds away from risk-free investments. As shown in table 1, the percentage of families holding stocks rose

[4] Fluctuations in stock prices were more pronounced in the late 1990s and those in house prices were greater in the early 2000s. However, I refer to the upward trend in stock and house prices over the entire decade as a unique bubbly episode, without distinguishing the "dot-com" bubble from the housing bubble. I can do that, because I just focus on the common features of the two bubbly episodes to explain the effect of bubbles on interest rates, and on net worth and consumption across generations.

Table 1: Assets

	Families holding asset (%)					
	1992	**1995**	**1998**	**2001**	**2004**	**2007**
Stocks	36.9	40.5	48.9	53	50.3	53.2
Certificates of deposit	16.7	14.3	15.3	15.7	12.7	16.1
Savings bonds	22.3	22.8	19.3	16.7	17.6	14.9

Source: SCF Chartbook 2016.

Note: Direct and indirect stock holdings are considered. Indirectly held stocks are those in mutual funds, retirement accounts, and other managed assets.

Table 2: Liabilities

	Type of debt (thousands of 2016$)					
	1992	**1995**	**1998**	**2001**	**2004**	**2007**
Total	72.2	76.2	93.5	98.2	131.5	145.9
Home-secured	97.5	101.3	114.7	124.2	157.8	172.6

Source: SCF Chartbook 2016.

Note: The average total debt is computed among those families holding some debt, while the average home-secured debt is computed among only those families holding this specific debt type.

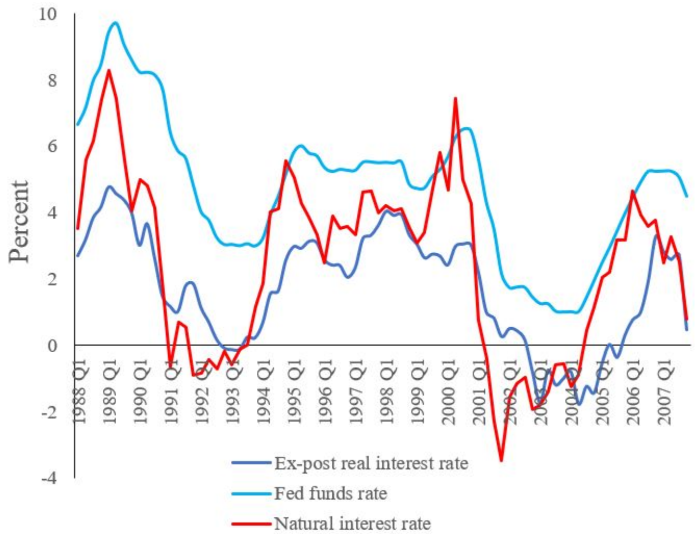
steadily over the different editions of the Survey of Consumer Finances (SCF), while the percentage of families with certificates of deposit was roughly stable and that of families with savings bonds strongly decreased.[5] The SCF also reports a large variation in the share of the value of families' financial assets attributed to (direct and indirect) stock holdings, which passed from 40% to 56% in the period 1995-2007. As a proportion of all families' assets, the value of primary and other residences instead increased from 35% to 39% in the same time span.[6] As regards borrowing behavior, table 2 shows a steady increase in the average value of families' total debt. Among the different types of debt contracts, home-secured debt stands out, because its value skyrocketed in the period under investigation. The build-up of home-secured signals borrowing against home equity, which was the main driver of the credit cycle (Justiniano et al., 2015).[7]

[5] The definition of "families" in the SCF is similar to that of "households" in the US Census Bureau. Certificates of deposit and savings bonds are categories of assets with a low risk profile. For a proper definition of these two categories, as well as for that of "families", see Bricker et al. (2017).

[6] The idea that houses served as investment vehicle is corroborated by Mian and Sufi (2018), who find that the rise in property transactions between 2003 and 2006 was due to speculative reasons.

[7] The average total debt is lower than the average home-secured debt, because a different number of families is considered in the two measures. Borrowing against home equity can happen through specific forms of home-secured debt: first-lien and junior-lien mortgages, and home equity lines of credit. Bucks et al. (2009) prove that the US households borrowed against the house value by providing accurate data on these forms of home-secured debt.

Figure 1: US interest rates

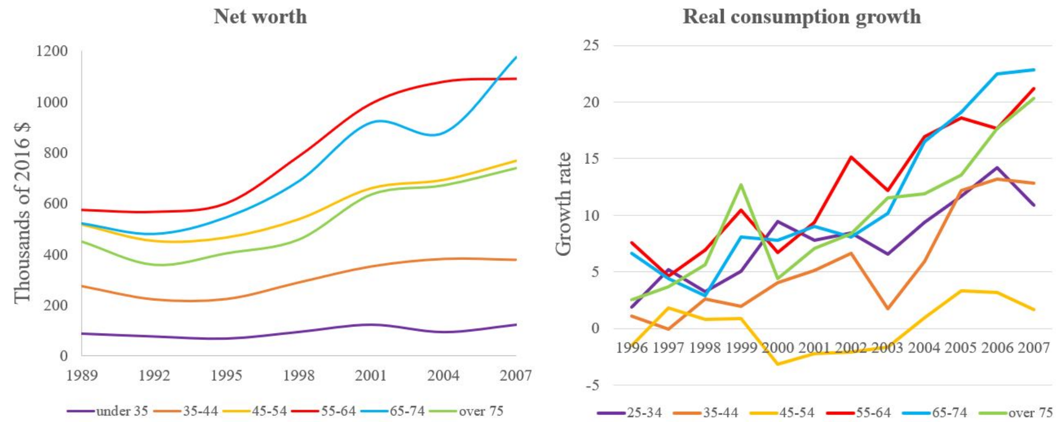


Source: FRED, Cúrdia (2015).
Note: Interest rates are at quarterly frequency. Federal funds rate and the annual growth rate of consumer price index (CPI) are used for the computation of the ex-post real interest rate.

Most of estimates of the natural rate of interest point to a less pronounced declining pattern in the decade preceding the global financial crisis. Laubach and Williams (2016) estimate the long-run real interest rate corresponding to potential output. Their measure fell consistently from 5% in the mid-1960s to 2% in the mid-1990s, and then it stabilized around 2-3% until the beginning of the Great Recession. The short-run natural interest rate computed by Cúrdia (2015) is plotted in Figure 1, along with nominal and real interest rates. This measure of the natural rate, though more volatile than its long-run counterpart, had a similar declining trend before 1994, while it fluctuated around 3-4% afterward. More precisely, the short-run natural rate recovered greatly a first time in the mid-1990s, when asset prices started to inflate, and it went back to their preexisting low level, after the stock market crash of 2000. Then, it went up again in correspondence of the new appreciation of assets observed in the early 2000s. This trend is shared with the federal funds rate and the ex-post real interest rate, despite they had different average values between 1995 and 2007.

Finally, net worth and real consumption grew unevenly across age cohorts. This is shown in Figure 2. The left panel of the figure depicts average net worth at constant prices (year 2016)

Figure 2: Net worth and real consumption growth across age cohorts



Source: SCF Chartbook 2016, Consumer Expenditure Survey.
Note: Consumption expenditure is expressed in real terms through the Consumer Price Index (CPI).

over the period 1989-2007, and the right panel depicts the growth rate of real consumption (1995 constant prices) relative to the base year 1995. Average net worth rose steeply for the middle-aged and old cohorts, especially for people between 55 and 74 years who experienced the largest wealth gains. The pattern of average net worth was instead flat for the young cohorts aged at most 44 years. Changes in net worth mostly reflected movements in asset prices (Aizcorbe et al., 2003; Bucks et al., 2009), which had a larger impact on the wealth of middle-aged and old cohorts. The young generation accumulated more debt than other generations anyway. In fact, the leverage ratio (families' total debt to total assets) of the age class 25-34 years was on average 36.6% in 1998 and it reached 44.3% in 2007, while the average leverage ratio of the class 35-44 years passed from 25.1% to 28.2% (Bucks et al., 2009). Real consumption growth followed closely the pattern of net worth. The age cohorts 55-64 years, 65-74 years and over 75 years reported the largest consumption growth rate in 2007 relative to the 1995 level. The pace of real consumption growth was slower for the young cohorts 25-34 and 35-44 years, and consumption grew even less for people between 45 and 54 years.

3 The Model Economy

I consider an OLG economy without capital which consists of households, firms and a central bank. Agents form expectations rationally. The generation born at t is composed by N_t agents and the ratio between the size of young and middle generations is $(1+g_t) = \frac{N_t}{N_{t-1}}$, where g_t is also the population growth rate.

Agents live for three periods, but they have a positive income only in middle age and so exchange financial assets in the credit and bubbles markets. The credit market works in the following way. Young households borrow to consume by issuing a one-period riskless bond, while funds are supplied by middle-aged households which save for retirement. After a period, borrowers get a positive income because middle-aged and repay debt to lenders, who have become old. As the young generation is limited in their ability to borrow, there are not sufficient opportunities for investment, and intrinsically worthless assets can be valued if they guarantee a higher return than bonds. Different varieties of these "bubbly" assets are traded in a proper market. Each period the middle generation creates a new variety of bubbles and buys from the old generation the varieties introduced by the previous cohorts, a fraction of which is destroyed. The old generation in turn initiated a new bubble the period before and purchased the other bubble varieties from the past generation.[8] The quantity of bubbly assets grows at the same rate as population. As bubbles allow agents to carry over funds to old age, they represent an investment vehicle. New bubbles also improve the ability to repay debt in middle age, so young households demand more funds in the credit market by using the future bubble as collateral.

Firms operate for just one period in a perfectly competitive market. As their number is equal to the size of the middle generation, the economy's growth rate is $(1+g_t)$. The labor market is perfectly competitive too, because workers and employers are wage takers. However, workers are unwilling to accept nominal wages lower than a minimum level (Schmitt-Grohé and Uribe, 2016). The downward wage rigidity makes monetary policy non-neutral and the gross real rate of return

[8] As pointed out by Martin and Ventura (2011) in a similar setting, a bubble is a claim on future savings, because it entitles the owner to receive a payment from the next generation. Middle-aged households issue directly this claim by initiating a new bubble, while they buy the claims issued by the past generations by purchasing old bubbles. For real-world examples of bubbly assets, see Martin and Ventura (2012).

from bonds has to satisfy the Fisher condition:

$$1 + r_t = (1 + i_t)\, E_t \Pi_{t+1}^{-1} \tag{1}$$

where i_t is the nominal interest rate, P_t is the price level, $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ is the gross inflation rate and E_t denotes the expectation operator.

I will describe the behavior of agents and I will study the bubbles and credit markets in this section.

3.1 Households

When young, households borrow to finance their consumption and face an exogenous debt limit D_t, which can be relaxed by using future bubbly assets as collaterals. Middle-aged households are endowed with a portion $\delta \in (0, 1)$ of a new bubbly asset whose price is $P^B_{t|t} \geq 0$, while a fraction δ of old bubbly assets is not longer traded. They also supply inelastically their labor endowment $\bar{L}$ for a wage W_t and run a firm whose profits are Z_t. The resulting income $Y_t = \frac{W_t}{P_t} L_t + \frac{Z_t}{P_t}$ is partially invested in risk-free bonds and different varieties of old bubbly assets.[9] All the proceeds from saving are consumed in old age. The representative household solves the problem:

$$\max_{C^m_{t+1}, C^o_{t+2}, Q^B_{t+1|t+1-j}} E_t \left\{ \ln C^y_t + \beta \ln C^m_{t+1} + \beta^2 \ln C^o_{t+2} \right\}$$

s.t.

$$C^y_t = B^y_t \tag{2}$$

$$C^m_{t+1} = Y_{t+1} + \delta P^B_{t+1|t+1} - (1 + r_t)\, B^y_t - B^m_{t+1} - \sum_{j=0}^{\infty} P^B_{t+1|t+1-j} Q^B_{t+1|t+1-j} \tag{3}$$

$$C^o_{t+2} = (1 + r_{t+1})\, B^m_{t+1} + (1 - \delta)\,(1 + g_t) \sum_{j=0}^{\infty} P^B_{t+2|t+1-j} Q^B_{t+1|t+1-j} \tag{4}$$

$$(1 + r_t)\, B^y_t = D_t + \delta E_t P^B_{t+1|t+1} \tag{5}$$

[9] Labor demand L_t can differ from labor supply $\bar{L}$, because the downward wage rigidity can determine labor rationing like in Schmitt-Grohé and Uribe (2016). I address this issue more in deep in the next paragraphs.

C_t^i is the consumption of each generation and B_t^i the real value of bonds with $i = y, m, o$. $Q^B_{t|t-j}$ and $P^B_{t|t-j}$ are the quantity at time t of the bubbly asset introduced by the cohort $t-j$ and its price. $Q^B_{t|t-j}P^B_{t|t-j}$ is accordingly the expenditure for the bubbly asset $t-j$ from the middle generation, whose total expenditure for all the varieties of bubbles is given by the summation in equation (3). The value of all bubbles purchased changes in the next period (equation (4)), when middle-aged households become old and sell their bubble holdings. In fact, the quantity of the bubbly asset $t-j$ grows at the rate g_t, though only a fraction $(1-\delta)$ of this bubble variety is still traded, while its price varies from $P^B_{t|t-j}$ to $P^B_{t+1|t-j}$. Equation (5) is the debt limit, which is binding by assumption for young households.[10] The optimality conditions for this problem are:

$$\frac{1}{C_t^m} = \beta\left(1 + r_t\right) E_t \frac{1}{C_{t+1}^o} \tag{6}$$

$$P^B_{t|t-j} = \left(1-\delta\right)\left(1+g_t\right)\beta E_t\left[\left(\frac{C_t^m}{C_{t+1}^o}\right)P^B_{t+1|t-j}\right] \tag{7}$$

Condition (6) is a standard Euler equation, while equation (7) expresses the market value at time t of the bubbly asset introduced in $t-j$. Bubbly asset has no fundamental value, but it is valued if the representative household expects to gain profit from selling it. Therefore the price of the bubble depends on its discounted expected value in the next period.

3.2 Firms

The technology of firms is described by the production function:

$$Y_t = L_t^{\alpha} \tag{8}$$

where L_t is the quantity of labor employed and $0 < \alpha < 1$. Taking prices as given, firms maximize their profit:

$$Z_t = P_t Y_t - W_t L_t \tag{9}$$

[10] This holds for $D_{t-1} < \frac{1}{1+(1+\beta)\beta}\left[Y_t - \beta\left(1+\beta\right)\delta P^B_{t|t}\right]$, a condition which is met in all the simulations presented below.

subject to (8). The resulting optimality condition is the labor demand:

$$\frac{W_t}{P_t} = \alpha L_t^{\alpha-1} \tag{10}$$

The downwardly rigid nominal wage can be expressed as:

$$W_t = \max\left[\gamma W_{t-1} + (1-\gamma) P_t \alpha \bar{L}^{\alpha-1}, P_t \alpha \bar{L}^{\alpha-1}\right] \tag{11}$$

The first term in the max operator denotes the minimum level of W_t workers are willing to accept, where $\gamma \in (0,1)$ is the degree of wage rigidity. This lower bound is the weighted average of the wage level in the last period and the "flexible" level compatible with full employment $P_t \alpha \bar{L}^{\alpha-1}$. When labor market clearing requires an increase in W_t from the last period, the nominal wage is flexible and there is full employment ($L_t = \bar{L}$). When the wage should be cut to maintain the full employment of resources, the downward rigidity prevents such price adjustment and involuntary unemployment arises ($L_t < \bar{L}$).

3.3 The Central Bank

The central bank behaves according to the interest rate rule:

$$1 + i_t = \max\left[1, \left(1 + r_t^f\right) \overline{\Pi} \left(\frac{\Pi_t}{\overline{\Pi}}\right)^{\phi_\pi}\right] \tag{12}$$

where $\phi_\pi > 1$, r_t^f is the natural interest rate and $\overline{\Pi}$ is the gross inflation target. The central bank maneuvers the nominal interest rate to track r_t^f, and it raises (cuts) the policy rate if inflation is higher (lower) than the target (Cúrdia et al., 2015). However, if the natural interest rate turns negative ($1 + r_t^f < 1$) and the targeted inflation rate is zero ($\overline{\Pi} = 1$), the central bank would set a negative nominal interest rate but it cannot because of the ZLB in equation (12).

3.4 Credit and Bubbles Markets

Bubbly assets market clearing requires:

$$Q^B_{t|t-j} = \delta\,(1-\delta)^j \tag{13}$$

The individual demand of the bubble variety $t-j$ at time t (aggregate demand is $N_{t-1}Q^B_{t|t-j}$) has to be equal to its supply, which depends on the quantity of the variety created by each middle-aged household in $t-j$ and on that which survives over time. Given the assumption on the endowment of new bubbles and those on destruction and growth rates of bubbles quantity, the total amount of bubbles in the economy is equal to the size of the middle generation. The economy's bubble index which includes new and old bubbles is:

$$P^B_t = \frac{\tilde{P}^B_t}{N_{t-1}} = \delta\sum_{j=0}^{\infty}(1-\delta)^j\,P^B_{t|t-j} \tag{14}$$

while the index for the old bubbles is:

$$B_t = \frac{\tilde{B}_t}{N_{t-1}} = \delta\sum_{j=1}^{\infty}(1-\delta)^j\,P^B_{t|t-j} \tag{15}$$

where $\tilde{P}^B_t = \delta N_{t-1}\sum_{j=0}^{\infty}(1-\delta)^j\,P^B_{t|t-j}$ and $\tilde{B}_t = \delta N_{t-1}\sum_{j=1}^{\infty}(1-\delta)^j\,P^B_{t|t-j}$. Both indexes are normalized in terms of the size of the middle generation N_{t-1}. The equation for the aggregate bubble index can be rewritten as:

$$P^B_t = U_t + B_t = (1+g_t)\,E_t\left[\frac{B_{t+1}}{(1+r_t)}\right] \tag{16}$$

by using equations (6), (7), (14) and (15). Equation (16), where $U_t = \delta P^B_{t|t}$ denotes the value of the new bubbly assets, is a no-arbitrage condition. New and old bubbly assets will be valued from rational agents in the next period, if their expected rate of return is the real interest rate. The term $(1+g_t)$ undoes the effect of growth in the quantity of bubbles on their total value.

The equilibrium in the credit market requires the amount of funds demanded equals that supplied, given the different size of borrowers (young households) and savers (middle-aged ones):

$$(1+g_t)\, B_t^y = B_t^m \tag{17}$$

Denote credit demand with D_t^c and credit supply with S_t^c. Substituting (5) into credit demand, we obtain:

$$D_t^c = \left(\frac{1+g_t}{1+r_t}\right)(D_t + E_t U_{t+1}) \tag{18}$$

Combining (3), (4), (5), (6) and (16) yields credit supply:[11]

$$\begin{aligned} S_t^c = B_t^m &= \frac{\beta}{1+\beta}(Y_t - D_{t-1} - U_t - B_t) - \frac{1}{1+\beta}(B_t + U_t) \\ &= \frac{\beta}{1+\beta}(Y_t - D_{t-1}) - (B_t + U_t) \end{aligned} \tag{19}$$

Equations (18) and (19) show the two channels through which bubbles affect saving and borrowing. First, bubbles operate through the *saving channel* by serving as store of value. An alternative investment vehicle diverts resources away from riskless bonds, as shown by the term $\frac{\beta}{1+\beta}B_t$ in equation (19). Equally it induces to save less by providing an additional income in old age (the term $\frac{1}{1+\beta}(B_t + U_t)$ in (19)). Second, bubbly assets serve as collateral, and they affect borrowing and saving through the *borrowing channel*. The bubbly collateral increases credit demand by $\left(\frac{1+g_t}{1+r_t}\right) E_t U_{t+1}$. A higher demand for credit from young households results in a higher debt to repay for middle-aged ones, and this decreases saving by $\frac{\beta}{1+\beta}U_t$ in equation (19).

The real interest rate which clears the credit market can be derived by equating (18) and (19):

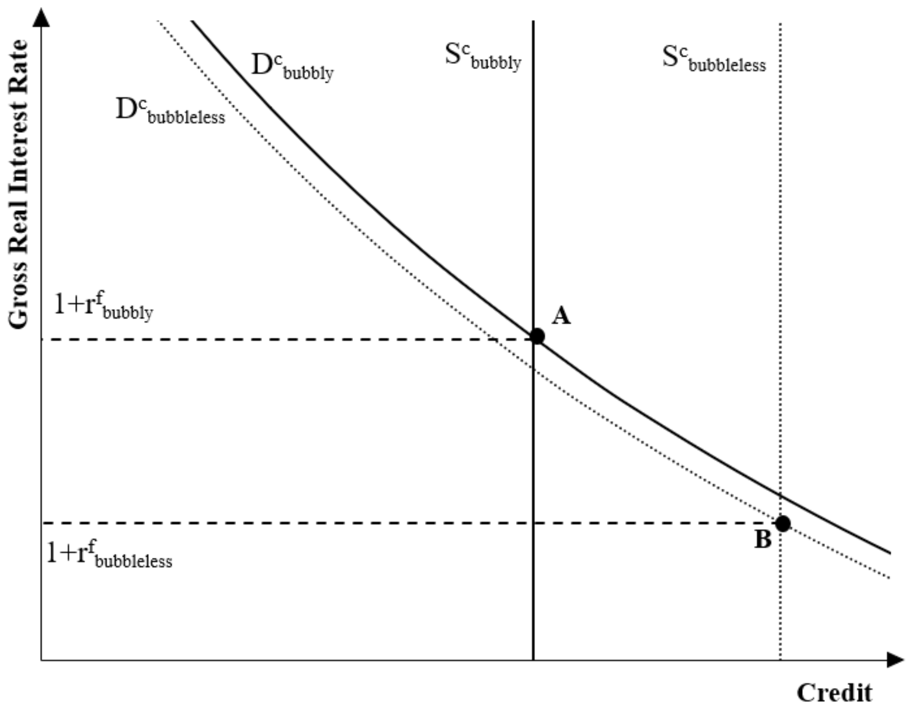
$$(1+r_t) = (1+g_t)\left[\frac{(1+\beta)(D_t + E_t U_{t+1})}{\beta(Y_t - D_{t-1} - U_t - B_t) - (B_t + U_t)}\right] \tag{20}$$

[11] We derive from (13), (14) and (15):

$$\sum_{j=0}^{\infty} P_{t|t-j}^B Q_{t|t-j}^B = U_t + B_t$$

$$(1-\delta)\sum_{j=0}^{\infty} P_{t+1|t-j}^B Q_{t|t-j}^B = E_t B_{t+1}$$

Figure 3: Equilibrium in the credit market



It corresponds to the natural interest rate at the potential level of production $Y^f = \bar{L}^{\alpha}$:

$$\left(1+r_t^f\right) = (1+g_t)\left[\frac{(1+\beta)\,(D_t + E_t U_{t+1})}{\beta\,(Y^f - D_{t-1} - U_t - B_t) - (B_t + U_t)}\right] \tag{21}$$

The way in which bubbles alter the natural rate of interest is depicted graphically in Figure 3, which plots the credit demand and supply curves in a bubbleless economy ($B_t = U_t = 0$) and in a bubbly one. Compared to a bubbleless economy, bubbly assets reduce credit supply through the saving and borrowing channels, and they foster demand for credit through the borrowing channel. As these two effects push r_t^f up by shifting the credit supply curve left and the credit demand curve right (Figure 3), the natural rate of interest is higher in a bubbly economy. This result will be crucial in the next sections, where I will study the effect of a permanent shock to r_t^f on the economy. In this case, a sufficiently large aggregate bubble can prevent the natural interest rate from turning negative, avoiding persistently low interest rates.

4 Steady State Equilibrium

A perfect foresight equilibrium is a set of quantities $\{C_t^y, C_t^m, C_t^o, B_t^y, B_t^m, Y_t, Z_t, L_t, B_t\}$ and prices $\{P_t, W_t, r_t, i_t\}$ that solve (1), (2), (3), (4), (5), (6), (8), (9), (10), (11), (12), (16) and (17), given $\{D_t, g_t, U_t\}$ and initial values for W_{-1}, B_{-1}^m and B_{-1}. I assume $g_t = g$, $D_t = D$ and $U_t = U$ with $g, D, U \geq 0$. Then, we get the law of motion of the old bubble by combining (16) and (21):

$$B_{t+1} = \frac{(1+\beta)(D+U)(U+B_t)}{\beta(Y^f - D - U - B_t) - (U+B_t)} = K(B_t, U) \tag{22}$$

A *bubbleless full employment steady state* (hereafter FE) corresponds to the pair $(B, U) = (0, 0)$ such that $B = K(0,0) = 0$, while a *bubbly full employment steady state* (FEB) is given by a pair (B, U) satisfying $B = K(B, U)$ with $B \in (0, Y^f)$. The FEB exists if:

$$D < \frac{\beta}{1+\beta}\left(Y^f - D\right) \tag{23}$$

and a formal proof for this necessary and sufficient condition is given in Appendix A. A too low debt limit prevents young households from issuing enough bonds to absorb all saving. This pushes the real interest rate below the economy's growth rate in a bubbleless economy, and so agents invest "rationally" in intrinsically worthless assets (Samuelson, 1958; Tirole, 1985). When condition (23) is satisfied, there exists a continuum of stable ($B^S(U), U$) and unstable ($B^U(U), U$) bubbly full employment steady states for any $U \in [0, \bar{U})$.[12] In what follows I restrict my attention to the stable FEB.

In the rest of the section, I compare the allocation of resources and the welfare implied by the FEB with those corresponding to the FE. I then analyze aggregate demand and supply in the two steady state equilibria to study qualitatively and quantitatively the mechanism through which bubbles counteract declining interest rates.

[12] Stable and unstable equilibria are depicted, along with the old bubble dynamics, in Figure 8 Appendix A. The stability of the FEB depends on the condition $\partial K(B,U)/\partial B < 1$, which also guarantees the stationarity of the old bubble, as proved in Appendix A. The existence, as well as the stationarity, of the bubble is verified in all the simulations reported in the paper. Finally, the condition $r < g$ has to be met even in a FEB, because the price of old bubbles grows at the rate r and, given the presence of new bubbles, the aggregate bubble would grow unboundedly if $g = r$. There is accordingly an upper bound on B like in Galì (2014). This is the value for which the real interest rate equals the growth rate of the economy, namely $B^U(0) = \frac{\beta}{1+\beta}\left(Y^f - D\right) - D$.

4.1 Redistributive Bubbles and Welfare

Here I study the allocation of consumption across generations in a FE and in a FEB. Bubbles do not affect the production of the economy, but just its allocation through the credit market. If bubbles are not valued and output is at the potential level, the steady state values of the main variables are:

$$
\begin{aligned}
(1+r_{FE}) &= \frac{(1+g)(1+\beta)D}{\beta(Y^f - D)} \\
B^y_{FE} &= \frac{D}{1+r_{FE}} \\
B^m_{FE} &= \frac{\beta}{1+\beta}\left(Y^f - D\right) \\
C^y_{FE} &= B^y_{FE} = \frac{1}{1+g}\left[\frac{\beta}{1+\beta}\left(Y^f - D\right)\right] \\
C^m_{FE} &= \frac{1}{1+\beta}\left(Y^f - D\right) \\
C^o_{FE} &= (1+g)D
\end{aligned}
$$

The first three equations have been derived in the last section and they are taken at the FE steady state equilibrium in which $B = U = 0$. The last three equations are computed from (2), (3) and (4) by substituting for $(1+r_{FE})$, B^y_{FE} and B^m_{FE}. The middle-aged agent saves a constant share of income net of debt, while the remaining share is consumed. The young generation receives savings in exchange for riskless bonds, but its size is larger than that of the middle generation. This reduces the amount B^y_{FE} each young household collects in the credit market and so the individual consumption C^y_{FE}. As one-period bonds are assets for the middle-aged households and liabilities for the young ones, the representative agent pays down the total amount of debt D to the elderly in middle age. Old households are fewer than middle-aged ones and this increases their proceeds from lending.

The same variables assume the following values in the FEB:

$$
\begin{aligned}
(1 + r_{FEB}) &= \frac{(1+g)(1+\beta)(D+U)}{\beta\left(Y^f - D - U - B\right) - (U+B)} \\
B^y_{FEB} &= \frac{D+U}{1+r_{FEB}} \\
B^m_{FEB} &= \frac{\beta}{1+\beta}\left(Y^f - D\right) - (U+B) \\
C^y_{FEB} &= \frac{D+U}{1+r_{FEB}} = \frac{1}{1+g}\left[\frac{\beta}{1+\beta}(Y^f - D) - (U+B)\right] \\
C^m_{FEB} &= \frac{1}{1+\beta}\left(Y^f - D\right) \\
C^o_{FEB} &= (1+g)(D+U+B)
\end{aligned}
$$

and they can be computed by following the same steps as above. The real interest rate is higher and less credit is supplied in a bubbly economy, while young households pledge new bubbles and so issue more bonds than in the FE. Bubbly collateral increases the debt repaid to old households (the term U in the last equation), whose consumption increases further because of their bubbly investment in middle age (the term B). On the other hand, an increased demand for credit does not necessarily cause a higher C^y in the FEB. C^y_{FEB} in contrast is lower than C^y_{FE} because of the lower credit supply, which decreases the funds borrowers raise in the credit market. Therefore young households demand more funds and pay down more debt when middle-aged, but they just have higher interest payments without raising more funds. The consumption of the middle-aged household is identical to that in the FE, because the positive and negative effects of bubbles cancel out:

$$
C^m_{FEB} = \frac{1}{1+\beta}\left(Y^f - D - U - B\right) + \frac{1}{1+\beta}(U+B) = \frac{1}{1+\beta}\left(Y^f - D\right)
$$

The additional income provided by the bubbly investment in old age changes the intertemporal allocation of consumption, because it induces middle-aged households to save less freeing up resources to consume ($\frac{U+B}{1+\beta}$). These resources are fully exhausted by the higher debt to repay and by the old bubble purchases.

The different allocation of consumption across generations implies distinct welfare levels in the two equilibria considered. Welfare is measured by the utility of the representative agent, which

I express as U_{FEB} in the FEB and as U_{FE} in the FE, and it is derived by substituting for the consumption of each generation into the utility function. The difference between U_{FEB} and U_{FE} is:

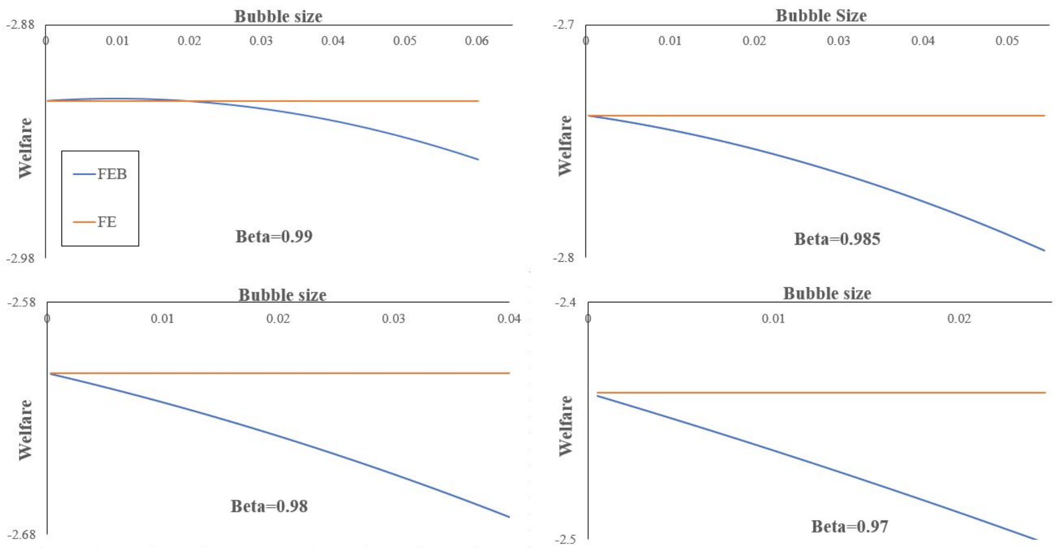
$$U_{FEB} - U_{FE} = \ln\left[1 - \frac{1+\beta}{\beta}\frac{(U+B)}{(Y^f - D)}\right] + \beta^2 \ln\left(1 + \frac{U+B}{D}\right) \tag{24}$$

It depends on the differential levels of C^y and C^o in the two steady states, which reflect the negative impact of bubbles on the consumption of young households (the first term on the right-hand side) and their positive effect on the consumption of old ones (the second term). The relative strength of these effects, which determines the sign of the difference in (24), is governed by β, D and the size of the aggregate bubble $P^B = U + B$. I now carry out two numerical exercises.

In the first one, I study how the difference $U_{FEB} - U_{FE}$ varies according to the bubble size for different calibrations of β. I set in this case $D = 0.22$, which is very close to the calibration of Eggertsson et al. (2019) in their quantitative model (0.234).[13] Results are presented in Figure 4, where the bubble size is plotted against the welfare level. U_{FE} is depicted by a horizontal line, because it is independent of the bubble size. The difference between U_{FEB} and U_{FE} is always negative, and the utility in the FEB declines as the bubble enlarges, widening the gap with the FE. The intuition underlying these results is straightforward. The consumption gains from a higher C^o in the FEB increase with the size of the bubble in equation (24), like the consumption losses from a lower C^y. U_{FEB} is lower than U_{FE}, because the losses are given a greater weight than the gains ($\beta < 1$). Furthermore, the marginal effect on U_{FEB} of a reduction in C^y increases with the size of the bubble, while the marginal effect on U_{FEB} of an increase in C^o declines. This explains why a larger bubble widens the negative gap between U_{FEB} and U_{FE}. A special case is represented by the calibration $\beta = 0.99$ which delivers $U_{FEB} > U_{FE}$ for small bubble sizes (top left panel of Figure 4). Smaller bubbles reduce the consumption losses in young age and the consumption gains in old age, but the negative effect on the size of consumption gains is compensated by a high β. This implies a great evaluation of the gains, as well as a high C^y which mitigates the marginal impact on

[13] I also set $\gamma = 0.98$, $\alpha = 0.7$, $\overline{\Pi} = 1$, $\phi_\pi = 2$ and $g = 0.023$, in order to get a full employment equilibrium in the bubbleless economy. The results hold for different calibrations of the parameters, as long as they deliver an equilibrium with full employment for $B = U = 0$. All the calibrated values in the paper are in annual terms and they have to be converted to 20 years, which is the length of a generation in my model like in that of Eggertsson et al. (2019). Finally, the maximum size of the bubble varies with β and D, because any change in these parameters alters the admissible range of values for B and U.

Figure 4: Bubble size and welfare for different values of β

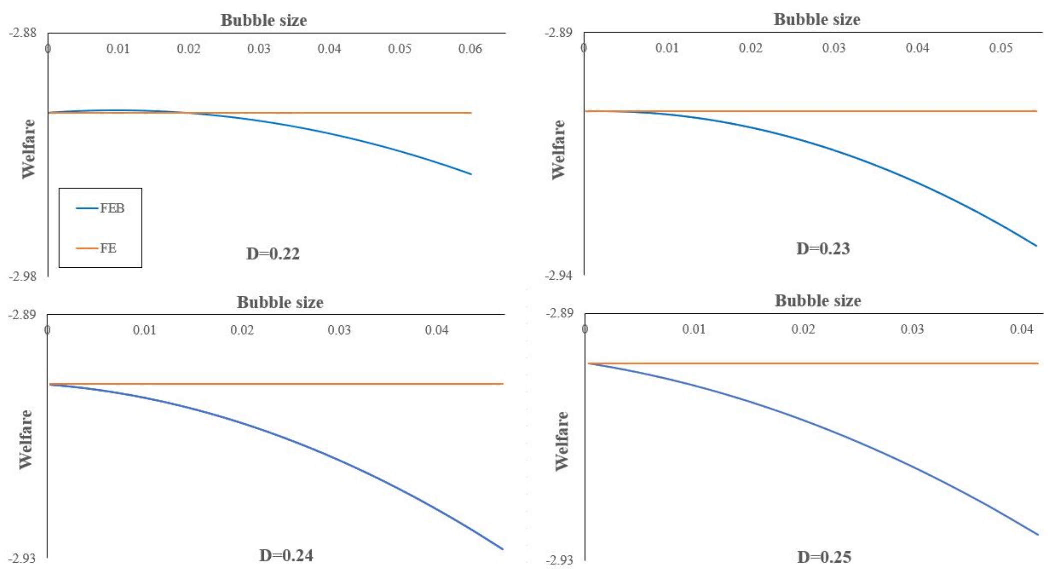


U_{FEB} of a reduction in the young age consumption (the term $\frac{1+\beta}{\beta}$ in equation (24) decreases).

I go through the same exercise, but I set $\beta = 0.99$ and consider different values of D. I check in this way the robustness of my results to variations in the debt limit, given that the chosen value of β is the only case for which $U_{FEB} > U_{FE}$ in the previous exercise. This numerical exercise is plotted in Figure 5, where the top left panel is identical to that in Figure 4. $U_{FEB} - U_{FE} < 0$ even for different calibrations of D, because a higher debt decreases savings of middle-aged households and so the consumption of young ones, and it increases the proceeds from lending of the elderly. This in turn amplifies the marginal effect on utility of consumption losses in youth and it dampens the marginal impact on utility of consumption gains in old age. The results of the second exercise not only corroborate those of the first one but reinforce them. As already mentioned, I have set in the first exercise a slightly lower D compared to Eggertsson et al. (2019) and U_{FEB} is not higher than U_{FE} anymore for calibrations of the debt limit closer to the benchmark model.[14]

[14] The range of values for D can be considered a plausible calibration of the debt limit, because the value set by Eggertsson et al. (2019) is exactly in the middle. I cannot set lower values anyway, given that they do not correspond to a bubbleless full employment equilibrium. Lower calibrations of β do not change the results. They in fact imply a lower consumption level in young age and this would amplify further the marginal effect on U_{FEB} of a reduction in C^y.

Figure 5: Bubble size and welfare for different values of D



4.2 How Bubbles Counteract Low Interest Rates

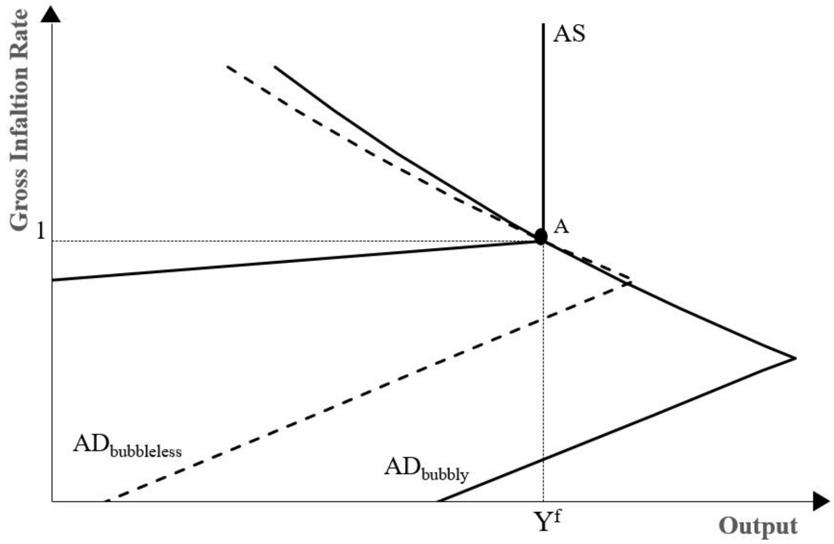
The steady state equilibrium can be represented by aggregate demand and supply. This alternative representation of the FEB and the FE is illustrated in this paragraph, because it allows to explain theoretically how asset price bubbles prevent low interest rates. I also measure the size of the bubble necessary to counteract declining interest rates, according to a standard calibration of the model.

4.2.1 Aggregate Demand and Supply

Aggregate demand and supply consist of two regimes. Aggregate supply (AS) is identical to that in Eggertsson et al. (2019), because bubbles only affect the demand-side of the economy. The regime of supply is determined by equation (11) through the inflation rate. If the steady state inflation is non-negative ($\Pi \geq 1$), the nominal wage is flexible and the equilibrium in the labor market is characterized by full employment. Supply corresponds to potential output in this case, and it can be computed from equations (8), (10) and (11):

$$Y_{AS} = \bar{L}^{\alpha} = Y^{f} \tag{25}$$

Figure 6: AD and AS in a bubbly and in a bubbleless economy



If inflation is negative ($\Pi < 1$), the nominal wage cannot fall enough to reach the level compatible with full employment because of the downward rigidity. The resulting involuntary unemployment determines a level of output below the potential, and AS, which can be derived by combining the same equations above, expresses a positive relation between inflation and output:

$$Y_{AS} = \left(\frac{1 - \gamma \Pi^{-1}}{1 - \gamma} \right)^{\frac{\alpha}{1-\alpha}} Y^f \tag{26}$$

Output increases when inflation goes up, because the real wage falls and firms produce more. Equations (25) and (26) are depicted respectively as a vertical and an upward sloping curve in Figure 6. The kink point at which the AS curve becomes upward sloping corresponds to the zero inflation level ($\Pi = 1$).

The regime of aggregate demand (AD) is governed by equation (12) which establishes whether the ZLB is binding or not. Combining equations (1), (12), and (20) yields the following AD with a positive policy rate ($1 + i_t > 1$):

$$Y_{AD} = D + \left(\frac{1+\beta}{\beta} \right) (U + B) + \left(\frac{1+\beta}{\beta} \right) (1 + g) \frac{\Gamma}{\Pi^{\phi_\pi - 1}} (D + U) \tag{27}$$

where $\Gamma = \overline{\Pi}^{\phi_\pi - 1}\left(1 + r^f\right)^{-1}$. If the nominal interest rate is zero ($1 + i_t = 1$), we get a different AD from the equations above:

$$Y_{AD} = D + \left(\frac{1+\beta}{\beta}\right)(U + B) + \left(\frac{1+\beta}{\beta}\right)(1+g)\,\Pi\,(D+U) \tag{28}$$

Demand is negatively related to inflation in equation (27) and it is represented by a downward sloping AD curve in Figure 6. Equation (28) in contrast relates positively aggregate demand and inflation, and it takes the shape of an upward sloping AD curve in the same figure. If the ZLB is not binding, the central bank raises more than proportionally the policy rate ($\phi > 1$) in response to an inflation increase. This pushes the real interest rate up, contracting demand and so stabilizing the inflation level. Ordinary monetary policy tools are inhibited in a liquidity trap, where the real interest rate depends only on the inflation level through the Fisher equation. Therefore an inflation increase decreases the real interest rate and expands demand.

The inflation level at which the central bank hits the ZLB in the attempt to stabilize inflation is depicted as a kink in the AD curve (Figure 6). This level of inflation, denoted by Π_{kink}, is computed by equating the two arguments in the right-hand side of (12), and expressing the resulting equation in terms of Π:

$$\Pi_{kink} = \left[\frac{1}{(1+r^f)}\right]^{\frac{1}{\phi_\pi}} \overline{\Pi}^{\frac{\phi_\pi - 1}{\phi_\pi}} \tag{29}$$

I plot in Figure 6,[15] together with the AD curve in a bubbly economy, the corresponding curve in a bubbleless economy ($U = B = 0$). The presence of bubbly assets does not change the nature of the full employment equilibrium, which occurs at the intersection of the vertical AS curve and the downward sloping AD curve anyway, and features $Y = Y^f$ and $\Pi = \overline{\Pi}$. Although both FEB and FE correspond to point A in the figure,[16] the underlying allocation of resources is different in the two steady states, like the natural level of the real interest rate. The higher natural interest rate characterizing the bubbly economy results in a lower Π_{kink} in equation (29) and so in a different location of the AD kink in Figure 6. This has fundamental implications when a negative shock to the natural rate of interest occurs, because it is less likely for the central bank to hit the ZLB for any

[15] I set $\beta = 0.987$, $\gamma = 0.98$, $\alpha = 0.7$, $\overline{\Pi} = 1$, $\phi_\pi = 2$, $D = 0.23$, $g = 0.023$ and $U = 0.0082$ to plot this figure.

[16] The standard Taylor principle ($\phi_\pi > 1$) guarantees the determinacy of FEB and FE, which are unique for a high enough γ and a low enough inflation target.

Table 3: Calibrated values

Parameters	Values	Description
β	0.987	Discount factor
α	0.7	Labor share
D	0.23	Collateral constraint
g	0.023	Population growth
$\overline{\Pi}$	1	Inflation target
ϕ_π	2	Taylor coefficient
γ	0.98	Wage rigidity
$\bar{L}$	1	Labor supply

given inflation level.

4.2.2 Calibrated Model

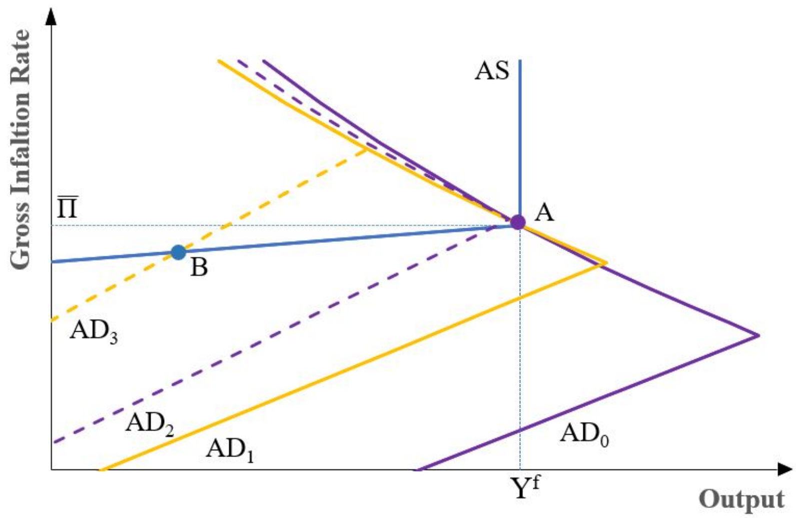
I perform here a calibration of the model by using US data. The aim of this exercise is not to measure the effect of asset price bubbles on interest rates. Rather, I want to show that the size of the bubble necessary to prevent low interest rates, when the natural interest rate falls, is reasonable according to a standard calibration of the model.

Table (3) contains the values assumed for the parameters of the model. The labor supply is equal to 1 to normalize all variables in terms of potential output. $\gamma = 0.98$ falls in the range of values found by Schmitt-Grohé and Uribe (2016), while D is set to approximately the value of the quantitative model of Eggertsson et al. (2019). The remaining parameters are standard and all are kept constant, except for g which goes from 0.023 to 0.01. I calibrate a demographic shock which leads the natural interest rate to -1% in a bubbleless economy. This value is consistent with the average real interest rate gap at annual level estimated by Cúrdia (2015) over the period 2009-2011.[17]

I plot Figure 7, which shows the response of a bubbleless and a bubbly economy to the calibrated shock, to clarify how bubbles avoid low interest rates. The initial equilibrium is given by point A, which is both a FEB and a FE. A permanent change in g decreases the fraction of young house-

[17] The nature of the shock which puts downward pressure on interest rates is not relevant for my theoretical and quantitative results. The shock assumed just simulates the effect on interest rates of several factors such as demography, technological and financial developments, income inequality, capital goods prices and global imbalances (Baldwin and Teulings, 2014; Summers, 2014). I choose the estimates of Cúrdia (2015) as a benchmark, because the theoretical model used for their computation incorporates a short-run definition of the natural interest rate which fits precisely with that in my model.

Figure 7: Demand shock in a bubbleless and in a bubbly economy



holds, reducing aggregate expenditure. The central bank cuts the nominal interest rate to stimulate consumption and in this way to compensate the drop in demand. If the economy starts from the FE, the monetary authority cannot keep demand at the potential level, when the natural rate of interest is negative and the inflation target is zero because of the ZLB. The resulting lack of demand creates deflationary pressures, so nominal rigidities are at work and wages cannot fall to clear the labor market. As a result, involuntary unemployment arises and output is below the potential.[18] The case of a bubbleless economy is represented by the yellow AD curves in Figure 7. The reduction in the natural interest rate lifts the AD kink (Π_{kink} increases) and the upward sloping demand curve shifts left from AD_1 to AD_3, which intersects aggregate supply in its upward sloping segment. The new equilibrium is B, which is characterized by low real and nominal interest rates because the ZLB is binding.[19]

The response of a bubbly economy to the same demand shock differs radically, as shown by the purple AD curves in Figure 7. As the natural interest rate is higher in the bubbly case, the downward

[18] As pointed out by Eggertsson et al. (2019), deflation is not crucial for this result. The mechanism outlined also works with positive inflation, as long as it is lower than the inflation target and nominal wages are indexed to a positive inflation target.

[19] Business cycle fluctuations around the steady state are possible, because an equilibrium such as B is determinate like the "secular stagnation" steady state in Eggertsson and Mehrotra (2014). The risk of more frequent and longer ZLB episodes associated with low interest rates is accordingly replicated.

sloping demand curve is longer (Π_{kink} is lower). So, even if the upward sloping AD curve shifts left from AD_0 to AD_2, the equilibrium is still determined at the intersection of the vertical AS curve in point A. This means that bubbles exerts an upward pressure, which prevents the natural interest rate from turning negative when a permanent demand shock occurs. As a consequence, the monetary authority can offset the shock via cuts in the policy rate, which is not constrained by the ZLB, and low interest rates do not appear. The minimum size of the aggregate bubble which is necessary to keep the natural rate of interest positive is approximately 0.05, which corresponds to 5% of total output ($Y = Y^f = 1$).[20] This figure proves that the existence of bubbly assets avoids low interest rates for a reasonable size of the bubble and a realistic calibration of the shock to the natural interest rate.

5 Conclusions

I have presented an OLG model consistent with the stylized facts characterizing the US economy before the Great Recession: the declining trend of interest rates slowed down between the mid-1990s and the mid-2000s, while the appreciation of stock and house prices altered saving and borrowing behaviors; and net worth and real consumption grew more in the older age cohorts than in the younger ones. My theoretical framework is able to account for these stylized facts because of the presence of rational asset price bubbles.

Bubbles absorb savings and reduce the propensity to save facilitating the transfer of resources to old age. They also serve as collaterals in the credit market fostering borrowing. These are respectively the *saving* and *borrowing channels*, that is the two mechanisms through which bubbles raise the natural interest rate. As a result of a higher natural interest rate, a structural negative shock to demand does not push the natural rate in negative territory, and so a central bank committed to price stability does not hit the ZLB. This result, which allows a bubbly economy to escape from low interest rates, holds for a realistic calibration of the demand shock and a reasonably large aggregate bubble. Bubbles also transfer resources from young age to old one and this makes the welfare of the representative agent worse than that in a bubbleless economy, for most of calibrations of the

[20] In the case of $1 + r^f = 1$, $\Pi_{kink} = \overline{\Pi} = 1$ as depicted in Figure 7. An alternative measure of the bubble could be the percent difference in the wealth of old households in the FEB and the FE. Wealth is 21.6 % higher in the FEB.

parameters and independently of the bubble size.

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Appendix

A Bubbly Full Employment Equilibrium

A.1 Existence

This proof is very close to that in Galì (2014, Appendix 2), so I refer to this paper for further details. K mapping has the following properties:

1. $K(B,U) \geq 0$ is twice continuously differentiable for $0 \leq B < \bar{B}(U)$, where $\bar{B}(U) = \frac{\beta}{1+\beta}\left(Y^f - D\right) - U$. If $B > \bar{B}(U)$, $K(B,U) < 0$.

2. The derivatives of $K(B,U)$ with respect to B_t are:

$$\frac{\partial K(B,U)}{\partial B_t} = \frac{\beta(1+\beta)(D+U)\left(Y^f - D\right)}{\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right]^2} > 0$$

$$\frac{\partial^2 K(B,U)}{\partial B_t^2} = \frac{2\beta(1+\beta)^2(D+U)\left(Y^f - D\right)}{\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right]^3} > 0$$

   The second inequality holds for $0 \leq B < \bar{B}(U)$ and $\lim_{B \to \bar{B}(U)} K(B,U) = +\infty$.

3. The derivatives of $K(B,U)$ with respect to U are:

$$\begin{aligned}\frac{\partial K(B,U)}{\partial U} =& \frac{(1+\beta)(D+2U+B)\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right]}{\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right]^2} + \\ & \frac{(1+\beta)^2(D+U)(U+B)}{\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right]^2} > 0\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 K(B,U)}{\partial U^2} =& \frac{2(1+\beta)\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right]^2}{\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right]^3} + \\ & \frac{2(1+\beta)^2(D+2U+B)\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right]}{\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right]^3} + \\ & \frac{2(1+\beta)^3(D+U)(U+B)}{\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right]^3} > 0\end{aligned}$$

   Both inequalities hold for $0 \leq B < \bar{B}(U)$ and $\lim_{B \to \bar{B}(U)} K(B,U) = +\infty$.

4. The mixed second derivative is:

$$\frac{\partial K(B,U)}{\partial B_t \partial U} = \frac{\beta(1+\beta)\left(Y^f - D\right)\left\{\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right] + 2(1+\beta)(D+U)\right\}}{\left[\beta\left(Y^f - D - U - B\right) - (U+B)\right]^3}$$

and it is positive for $0 \leq B < \bar{B}(U)$ and $\lim_{B \to \bar{B}(U)} K(B,U) = +\infty$.

Consider first the case $U = 0$. Equation (22) becomes:

$$B_{t+1} = \frac{(1+\beta) D B_t}{\beta\left(Y^f - D - B_t\right) - B_t} = K(B_t, 0)$$

A solution to this equation is the FE $(B,U) = (0,0)$. A FEB $(B^U, 0)$ with $B^U \in (0, Y^f)$ is another solution of the equation if:

$$\frac{\partial K(0,0)}{\partial B_t} = \frac{(1+\beta) D}{\beta\left(Y^f - D\right)} < 1$$

This condition, which is necessary and sufficient for the existence of the FEB $(B^U, 0)$, derives from property 2 and it can alternatively expressed as:

$$D < \frac{\beta}{1+\beta}\left(Y^f - D\right)$$

The FEB is unstable for the same reasons expressed in Galì (2014).

Sufficiency: Assume condition (23) holds. Given property 3 and the continuity of K, there are two steady states $B^U(U)$ and $B^S(U)$ for any $U \in (0, \bar{U})$ with $\bar{U} = \left[(1+2\beta) D - \beta Y^f\right]^2 / \left[4\beta(1+\beta)\left(Y^f - D\right)\right]$. $B^U(U)$ and $B^S(U)$ have the same stability properties of the equilibria in Galí (2014), and $B^U(U) > B^S(U)$. These two equilibria are depicted in Figure 8.

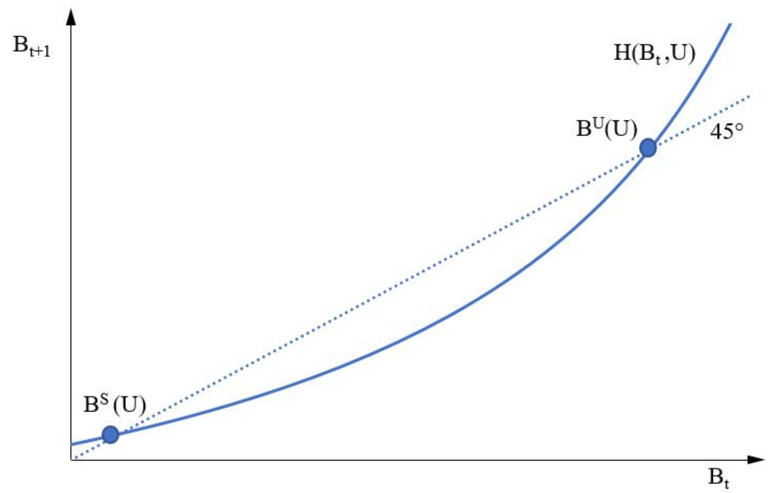
Necessity: The proof is equivalent to that in Galì (2014).

A.2 Stationarity of the old bubble

Taken at t, equation (22) becomes:

$$B_t = \frac{(1+\beta)(D_{t-1} + U_t)(U_{t-1} + B_{t-1})}{\beta\left(Y^f - D_{t-2} - U_{t-1} - B_{t-1}\right) - (U_{t-1} + B_{t-1})}$$

Figure 8: Bubble dynamics



Denoting log-linearized variables by lowercase letters, the log-linearized version of the equation above is:

$$b_t = \varphi v_b b_{t-1} + \psi_u u_t + \varphi v_u u_{t-1} + \psi_d d_{t-1} + v_d d_{t-2}$$

where $\varphi = \frac{\beta\left(Y^f-D\right)}{[\beta(Y^f-D-U-B)-(U+B)]}$, $v_b = \frac{B}{U+B}$, $\psi_u = \frac{U}{D+U}$, $v_u = \frac{U}{U+B}$, $\psi_d = \frac{D}{D+U}$, and $v_d = \frac{\beta D}{[\beta(Y^f-D-U-B)-(U+B)]}$. The condition for the stationarity of the old bubble is:

$$\varphi v_b = \left[\frac{\beta\left(Y^f-D\right)}{\beta\left(Y^f-D-U-B\right)-(U+B)}\right]\left(\frac{B}{U+B}\right) < 1$$

and it coincides with that for the stability of the FEB:

$$\frac{\partial K\left(B_t,U\right)}{\partial B} = \left[\frac{\beta\left(Y^f-D\right)}{\beta\left(Y^f-D-U-B\right)-(U+B)}\right]\left[\frac{(1+\beta)\left(D+U\right)}{\beta\left(Y^f-D-U-B\right)-(U+B)}\right] < 1$$

Indeed, it directly follows from equation (22) taken at the FEB:

$$\left(\frac{B}{U+B}\right) = \left[\frac{(1+\beta)\left(D+U\right)}{\beta\left(Y^f-D-U-B\right)-(U+B)}\right]$$

and so:

$$\frac{\partial K(B,U)}{\partial B} = \left[\frac{\beta\left(Y^f - D\right)}{\beta\left(Y^f - D - U - B\right) - (U + B)}\right]\left(\frac{B}{U+B}\right) = \varphi v_b$$