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Backward-looking and forward-looking notional-defined-contribution pension schemes[♠]

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Abstract

In order to spread notional capital accrued at retirement by members of a cohort over life expectancy, pay-as-you-go notional-defined-contribution (payg-ndc) scheme uses multipliers (different by retirement age) called conversion coefficients. These are backward-looking (b.l.) in that they relay on survival rates observed for previous cohorts in the past. Under increasing longevity, b.l. coefficients undervalue life expectancy, thus preventing full implementation of actuarial fairness (benefits equivalent to contributions) which is the main objective of ndc scheme. They also engender chronic deficits.

Forward-looking (f.l.) coefficients, relaying on survival rates forecast for the cohort whom coefficients themselves are assigned to, can improve actuarial fairness. Nevertheless, they face a rather serious political difficulty in that forecasting tools are fallible. This explains why switching to f.l. coefficients is unable to gain social consensus.

Abstracting from this, the paper shows that forward-looking coefficients produce ‘overshooting’. In fact, they generate chronic surpluses. The paper also shows that frontloading pension profile helps sustainability because it reduces both surpluses and deficits generated, respectively, by f.l. and b.l. approaches.

Introduction

A pay-as-you-go notional-defined-contribution (payg-ndc) pension scheme, rewarding contributions according to an interest rate equal to wage bill growth rate, is sustainable (produces an expenditure equal to tax revenue generated by a fixed tax rate) under a set of sufficient conditions. These include:

1) *exponential demography*, i.e. each cohort (starting employment at time t) must exceed its predecessor (starting employment at $t-1$) according to a percentage not varying over time; and
2) *constant longevity*, i.e. the rates according to which individuals survive from one age to the next, must be the same for all cohorts.[1]

Contrary to consensus of previous literature,[2] Gronchi and Nisticò (2008) proved that exponential growth of the average wage is not necessary for sustainability.[3]

They also proved that relaxing exponential demography, while keeping constant longevity, does not guarantee permanent (each year) sustainability any longer. Nevertheless, payg-ndc scheme can automatically reabsorb, with a lag, unbalances caused by shocks to cohort growth (shifts from one exponential path to another).[4]

To summarize, it is now clear that, under constant longevity, payg-ndc scheme is sustainable in two aspects: (i) if demography is rigorously exponential, the scheme is permanently (each year) balanced; (ii) it can at least generate a tendency to balance otherwise.

This paper is concerned with the effects of relaxing constant longevity while keeping exponential demography.

1 The structure of payg-ndc schemes

Ndc scheme is a payg system where each individual has a (notional) personal account where contributions are 'deposited' and benefits 'withdrawn'. All accounts (of active workers and pensioners alike) earn the same rate of interest conventionally chosen by the scheme. Such a rate may be constant or else indexed to either real economic variables (e.g. inflation possibly marked-up, productivity growth, wages growth, GDP growth) or financial variables (e.g. public debt average cost, return to a basket of securities). The 'withdrawal plan' - the calendar of pension installments – is designed to

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[1] It is also necessary that agents, if they are heterogeneous, i.e. marked by different behaviors (wage curves and/or retirement ages) be 'steadly' distributed (according to constant frequencies) among behaviors themselves.

[2] See, for instance, Valdes Prieto (2000).

[3] Exponential growth of the average wage still must be matched with exponential demography and constant longevity for a payg defined-benefit scheme permanently adopting the 'equilibrium' tax rate (which produces a revenue equal to expenditure) to guarantee a rate of return on contributions equal to wage bill growth rate . In Gronchi-Nisticò (2008) is pointed out the 'generational' nature of such a return, which bears on the entire mass of contributions paid in by a cohort, not individual contributions of single members (unless the latter are clones).

[4] Gronchi-Nisticò (2008).

guarantee actuarial fairness, i.e. that, on the average, deposits are exhausted, hence contributions plus interests are returned.

We shall discuss the nature and properties of a payg-ndc scheme under a set of non-distorting, simplifying hypotheses. These are the following:

- time is discrete even though its measurement unit is called as 'year';
- the first year of work is the first year of life;
- all individuals retire (perceive the first pension installment) at the same age[5];
- individuals never die before retirement, i.e. annual survival rates are equal to one until then;
- after retirement, survival frequencies decrease with age;
- survival benefits are excluded.

The following nomenclature $n, m,\ c_a,\ p_a,\ \hat{s}_a,\ d_a,\ \pi_a$ denotes, respectively: *n,* length of contribution period; *m,* maximum duration of the pension; c_a contribution paid in by an individual at age *a*; p_a, pension installment perceived at age *a;* $\hat{s}_a$, probability of surviving from retirement to age *a*; d_a, notional deposit held by an individual (worker or pensioner) at age *a;* π_a, value the interest rate assumed by the scheme takes when the individual matures age *a*.

Note that $n+1$ is the retirement age while $n+m$ is the impassable age limit (maximum possible age).

Probability $\hat{s}_a$ is an *ex-ante* rate assumed by the scheme that must be distinguished from *ex-post* rate, denoted as s_a, according to which individuals actually survive until age *a*, hence perceive installment p_a. One can say that probability $\hat{s}_a$ is 'perfectly imputed' when $\hat{s}_a = s_a$.

1.1 Actuarial fairness

The notional deposit held by an individual just before he perceives the first pension instalment takes the following form

$$(1) \qquad K = \sum_{a=1}^{n} c_a \cdot \prod_{a=a+1}^{n+1} \left(1+\pi_a\right) ,$$

and it is called the notional capital accrued at retirement. After the first annual instalment has been paid, the deposit balance changes to

$$(2) \qquad d_{n+1} = K - p_{n+1}$$

and next it is expected to follow the time path

$$(3) \qquad d_a = d_{a\text{-}1}\left(1+\pi_a\right) - \hat{s}_a \cdot p_a \ , \ a = n+2, \ldots n+m \, .$$

From (2) and (3), one can derive the final deposit (expected at age $n+m$) as follows:

---

[5] Most payg-ndc schemes allow for flexible retirement within a set age interval.

$$
(4) \quad \hat{d}_{n+m}(\boldsymbol{\pi},\mathbf{p}) = \underbrace{K \prod_{\alpha=n+2}^{n+m} (1+\pi_\alpha)}_{\substack{\text{future value of}\\ \text{notional capital}}} + \\
-\underbrace{\left[p_{n+1} \prod_{\alpha=n+2}^{n+m} (1+\pi_\alpha) + \sum_{\alpha=n+2}^{n+m-1} \hat{s}_a \cdot p_a \prod_{\alpha=a+1}^{n+m} (1+\pi_\alpha) + \hat{s}_{n+m} \cdot p_{n+m} \right]}_{\text{future value of benefits}} .
$$

On the left side, $\boldsymbol{\pi}$ denotes the vector $[\pi_{n+2},\dots,\pi_{n+m}]$ of order $m-1$, henceforth the 'interest profile', while $\mathbf{p}$ denotes the vector $[p_{n+1},\dots,p_{n+m}]$ of order m, henceforth the 'pension profile'.

Actuarial fairness is imposed with the following equation:

$$
(5) \qquad d_{n+m}(\boldsymbol{\pi},\mathbf{p}) = 0 \quad ,
$$

where unknowns are as many as $2 \times m - 1$.

Equation (5) associates an infinite set of actuarially fair pension profiles with each interest profile. Such pension profiles can only be determined once the time has elapsed to let the interest profile be fully known – hence beyond the maximum age limit. This raises the problem (well known to actuaries) of how $\mathbf{p}$ can be designed 'ahead of' $\boldsymbol{\pi}$. The ndc scheme adopts the following algorithm:

$$
(6) \qquad p(\boldsymbol{\pi},\delta) = \begin{cases} p_a = \underbrace{\dfrac{1+\pi_a}{1+\delta}}_{\substack{\text{indexation}\\ \text{factor}}} \cdot p_{a-1} \ , a = n+2,\dots,n+m \\ p_{n+1} = K \cdot \underbrace{\left[1 + \sum_{a=n+2}^{n+m} \hat{s}_a (1+\delta)^{n+1-a} \right]^{-1}}_{\text{conversion coefficient}} , \end{cases}
$$

where

$$
(7) \qquad \delta \in [0,\infty)
$$

is a constant rate arbitrarily chosen (by the scheme).[6] Algorithm (6) indexes the pension to the interest rate π 'diminished' by δ and calculates the first pension installment by multiplying the notional capital accrued at retirement by the conversion coefficient within square brackets.

Note that algorithm calculates first installment regardless of all elements of $\boldsymbol{\pi}$, none of which has still occurred at the time of retirement. Also note that each successive installment p_a is calculated on elements $\pi_{n+2},\dots,\pi_a$ which have already occur at age a. Therefore, algorithm overcomes the *a priori* lack of knowledge of $\boldsymbol{\pi}$.

One can now prove the following

[6] In principle, δ can be selected from wider interval $(-1,\infty)$.

Proposition 1: $d_{n+m}\left[\boldsymbol{\pi}, p\left(\boldsymbol{\pi},\delta\right)\right]=0\ \forall\boldsymbol{\pi}$ [pension profile (6) is actuarially fair whatever realization of $\boldsymbol{\pi}$].

Proof. Substituting algorithm (6) into constraint (4) one obtains

$$d_{n+m}\left[\boldsymbol{\pi}, p\left(\boldsymbol{\pi},\delta\right)\right]=K\prod_{a=n+2}^{n+m}\left(1+\pi_a\right)-\underbrace{\underbrace{K\left[1+\sum_{a=n+2}^{n+m}\hat{s}_a\left(1+\delta\right)^{n+1-a}\right]^{-1}}_{p_{n+1}}\left\{\prod_{a=n+2}^{n+m}\left(1+\pi_a\right)+\sum_{a=n+2}^{n+m}\frac{\hat{s}_a\cdot\prod_{a=n+2}^{n+m}\left(1+\pi_e\right)}{\left(1+\delta\right)^{a-n-1}}\right\}}_{\textit{future value of benefiys}}$$

$$=\prod_{a=n+2}^{n+m}\left(1+\pi_a\right)\left\{K-K\left[1+\sum_{a=n+2}^{n+m}\hat{s}_a\left(1+\delta\right)^{n+1-a}\right]^{-1}\left[1+\sum_{a=n+2}^{n+m}\hat{s}_a\cdot\left(1+\delta\right)^{n+1-a}\right]\right\}$$

$$=0\,,$$

QED.

1.2 Pension profile

Algorithm (6) associates pension profiles **p**, depending on δ, to any given interest profile $\boldsymbol{\pi}$. The differences among pension profiles, corresponding to different δ values, emerge from examination of the following two functions:

$$(8)\qquad h_a\left(\delta\right)=\frac{1+\pi_a}{1+\delta}\quad,$$

which is the indexation factor, and

$$(9)\qquad k\left(\delta\right)=\left[1+\sum_{a=n+2}^{n+m}\hat{s}_a\left(1+\delta\right)^{n+1-a}\right]^{-1}\quad,$$

which is the conversion coefficient. Both of them contribute to algorithm (6).

Figures 1 and 2 show the graphs. The following conclusions (see Figure 3) can be set: the higher δ, the more 'front-loaded' the pension profile is (i.e. the more quickly the scheme returns contributions). The most front-loaded profile is generated when $\delta=\infty$. The indexation factor becomes null and the conversion coefficient becomes equal to unity. The pension profile thus reduces to a lump sum equal to the notional capital accrued at retirement. The least front-loaded profile is generated when $\delta=0$. The indexation factor equals the interest factor while the first pension installment equals notional capital accrued at retirement divided by the sum of the m probabilities according to which installments are expected to be perceived (whose first equals unity).

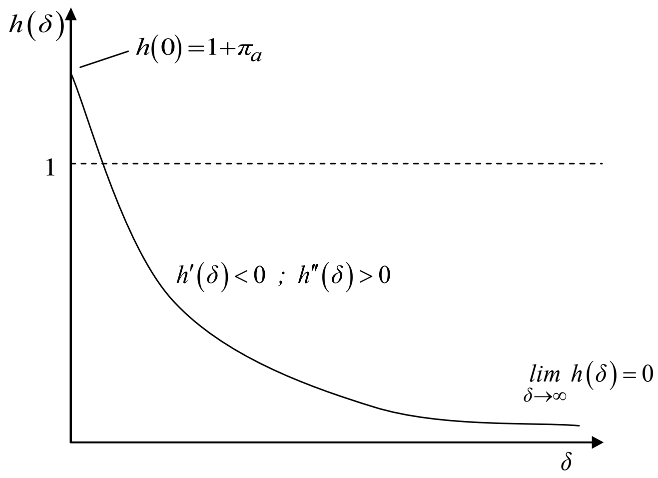

Fig. 1

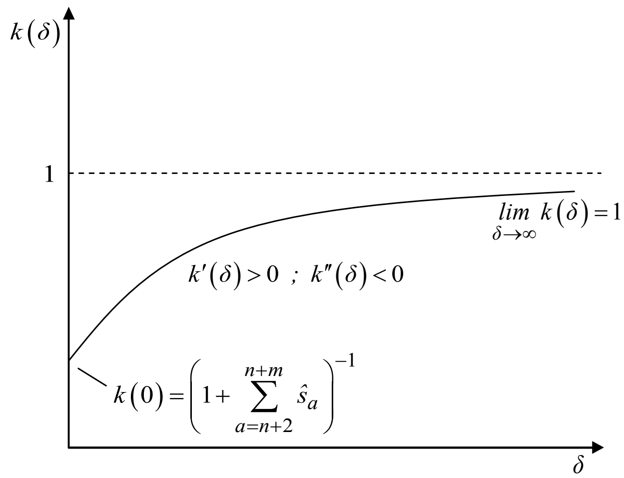

Fig. 2

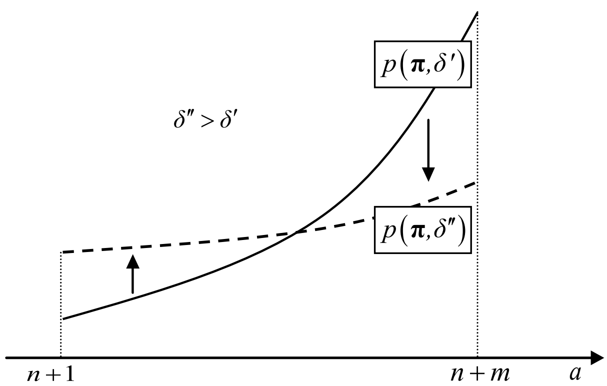

Fig. 3

2 Sustainability under constant longevity

Of the two conditions for sustainability mentioned in the introductory section, exponential demography and constant longevity, this paper is only concerned with the latter. The former can be 'left out' of the analysis by assuming $n=1$ (contribution period is limited to one year). In fact, under such assumption one can prove that constant longevity alone can guarantee sustainability. More precisely, denoted

- the rate of growth of the wage bill as ω,
- the table (vector) of 'dated' survival probabilities $\left[\hat{s}_{\tau,n+2},\dots,\hat{s}_{\tau,n+m}\right]$ imputed to cohort retiring in year τ, as $\hat{\mathbf{s}}_\tau$,
- the table (vector) of *ex-post* survival rates $\left[s_{\tau,n+2},\dots,s_{\tau,n+m}\right]$ experimented by the same cohorts, as $\mathbf{s}_\tau$,
- the balance of the scheme in year *t* as B_τ,

one can prove the following

Proposition 2: in a $n=1$ type payg-ndc scheme rewarding contributions at $\pi=\omega$ rate:

$$\hat{\mathbf{s}}_t=\mathbf{s}_t=\mathbf{s}_{t-1}\ ,\forall t \Rightarrow\ B_t=0\ ,\forall t$$

(perfectly imputed, constant longevity produces balances).

Proof. In case $m=2$, when one cohort of workers overlaps two of pensioners, expenditure in year *t* takes the form

$$E_t=\underbrace{\overbrace{\frac{a\,W_t\left(1+\pi_t\right)}{1+\omega_t}}^{\substack{\text{notional capital}\\ \text{accrued by cohort}\\ \text{retiring in year } t}}\cdot\overbrace{\frac{1+\delta}{1+\delta+\hat{s}_{t,3}}}^{\substack{\text{conversion}\\ \text{coefficient}\\ \text{in yeat } t}}}_{\text{pensions starting in year } t}+\underbrace{s_{t-1,n+2}\cdot\overbrace{\frac{a\,W_t\left(1+\pi_{t-1}\right)}{\left(1+\omega_{t-1}\right)\left(1+\omega_t\right)}}^{\substack{\text{notional capital}\\ \text{accrued by cohort}\\ \text{retiring in year } t\text{-}1}}\cdot\overbrace{\frac{1+\delta}{1+\delta+\hat{s}_{t-1,3}}}^{\substack{\text{conversion}\\ \text{coefficient}\\ \text{in year } t\text{-}1}}\cdot\overbrace{\frac{1+\pi_t}{1+\delta}}^{\substack{\text{indexation}\\ \text{factor}\\ \text{in year } t}}}_{\text{pensions starting in year } t\text{-1 surviving to year } t}\ ,$$

where W_t denotes the wage bill in year *t*. Note that conversion coefficients are 'dated', i.e. related to year of retirement. Since contribution revenue is $a\cdot W_t$, the balance (difference between revenue and expenditure) takes the form

$$(10)\qquad B_t=a\,W_t\left\{1-\left[\frac{1+\delta}{1+\delta+\hat{s}_{t,3}}\cdot\frac{1+\pi_t}{1+\omega_t}+\frac{s_{t-1,3}}{1+\delta+\hat{s}_{t-1,3}}\cdot\frac{\left(1+\pi_{t-1}\right)\left(1+\pi_t\right)}{\left(1+\omega_{t-1}\right)\left(1+\omega_t\right)}\right]\right\}.$$

Since present proposition assumes $\pi_\tau=\omega_\tau\ ,\ \tau=t-1,t$, the balance con be simplified as follows:

$$(11)\qquad B_t=a\,W_t\left\{1-\left[\frac{1+\delta}{1+\delta+\hat{s}_{t,3}}+\frac{s_{t-1,3}}{1+\delta+\hat{s}_{t-1,3}}\right]\right\}.$$

Hence $B_t=0$ if and only if the term within square brackets equals unity. This is true if $\hat{s}_{t,3}=\hat{s}_{t-1,3}=s_{t-1,3}$ (longevity is constant and perfectly imputed to cohorts). Extending the proof to the case when $m>2$ only involves formal complications. **QED**.

We shall keep $n=1$ hypothesis throughout the remaining part of the paper as equivalent to exponential demography.

2.1 Sustainability versus debt rollover

Consider the case when $m=2$. The contribution revenue of year t creates obligations to the ndc scheme which must be honored in years $t+1$ and $t+2$. The present value of these obligations is the same of a debt issued by the system in year *t*. Denoted the contribution revenue of year *t* as R_t, the debt issued in year *t* takes the following form

$$D_t^i = \underbrace{\overbrace{R_t\left(1+\pi_{t+1}\right)}^{\text{notional capital}} \cdot \overbrace{\frac{1+\delta}{1+\delta+\hat{s}_{t+1,3}}}^{\text{conversion coefficient}} \cdot \overbrace{\left(1+\pi_{t+1}\right)^{-1}}^{\text{discounting factor}}}_{\text{present value of pension benefits to be paid in year } t+1} +$$

$$+ \underbrace{\overbrace{s_{t+1,3}}^{\text{survival rate}} \cdot \overbrace{R_t\left(1+\pi_{t+1}\right)}^{\text{notional capital}} \cdot \overbrace{\frac{1+\delta}{1+\delta+\hat{s}_{t+1,3}}}^{\text{conversion coefficient}} \cdot \overbrace{\frac{1+\pi_{t+2}}{1+\delta}}^{\text{indexation factor}} \cdot \overbrace{\left[\left(1+\pi_{t+1}\right)\left(1+\pi_{t+2}\right)\right]^{-1}}^{\text{discounting factor}}}_{\text{present value of pension benefits to be paid in year } t+2},$$

which can be simplified as follows

$$(12) \qquad D_t^i = R_t\left(\frac{1+\delta}{1+\delta+\hat{s}_{t+1,3}} + \frac{s_{t+1,3}}{1+\delta+\hat{s}_{t+1,3}}\right).$$

If longevity is perfectly imputed, $\hat{s}_{t+1,3} = s_{t+1,3}$ such that equation (12) becomes $D_t^i = R_t$. On the other hand, E_t is interpretable as the portion of the existing debt (issued in years $t-1$ and $t-2$) which is repaid in year *t*. Denoted such a portion as D_t^r, $B_t = 0$ is the same as $D_t^i = D_t^r$.

Therefore, one can reformulate Proposition 2 as follows:

$$\hat{\mathbf{s}}_t = \mathbf{s}_t = \mathbf{s}_{t-1}, \forall t \Rightarrow D_t^e = D_t^r, \forall t$$

(debt can be rolled over if longevity is constant and perfectly imputed to cohorts).

As an obvious corollary, each year’s debt is the previous year’s one augmented at π rate, hence the debt increases exponentially at the same rate as the wage bill.

2.2 Unsustainable rates of interest

Constant and perfectly imputed longevity is a sufficient condition for sustainability of $n=1$ type payg-ndc scheme rewarding contributions at the wage bill growth rate. Effects of other rates are explored by the following

Proposition 3: in a $n=1$ type payg-ndc scheme, rewarding contributions at a $\pi \neq \omega$ rate,

$$\hat{\mathbf{s}}_t = \mathbf{s}_t = \mathbf{s}_{t-1}\,,\ \forall t \Rightarrow \begin{cases} B_t < 0 \\ \dfrac{\partial B_t}{\partial \delta} > 0 \end{cases},\ \forall t \ \text{ if } \ \pi > \omega$$

$$\hat{\mathbf{s}}_t = \mathbf{s}_t = \mathbf{s}_{t-1}\,,\ \forall t \Rightarrow \begin{cases} B_t > 0 \\ \dfrac{\partial B_t}{\partial \delta} < 0 \end{cases},\ \forall t \ \text{ if } \ \pi < \omega$$

(perfectly imputed, constant longevity generates deficits if contributions are rewarded at a higher rate than the wage bill growth rate, while it generates surpluses otherwise. In both cases disequilibria can be limited by increasing δ).

Proof. In case $m = 2$, perfectly imputed constant longevity implies $\hat{s}_{t,3} = \hat{s}_{t-1,3} = s_{t-1,3}$ at the right side of equation (10). Hence the term in square brackets is greater than 1 if $\pi_\tau > \omega_\tau\ (\tau = t-1, t)$ and smaller if $\pi_\tau < \omega_\tau\ (\tau = t-1, t)$. In both cases the sign of B_t follows. To prove the remaining part of the proposition, let us considere the following derivative of (10) with respect to δ:

$$(13) \qquad \frac{\partial B_t}{\partial \delta} = \frac{a\, W_t\, s_3}{1+\delta+s_3} \cdot \frac{1+\pi_t}{1+\omega_t} \cdot \left(\frac{1+\pi_{t-1}}{1+\omega_{t-1}} - 1 \right),$$

where s_3 denotes the common value of $\hat{s}_{t,3}, \hat{s}_{t-1,3}, s_{t-1,3}$. Extending the proof to the case when $m > 2$ only involves formal complications. **QED**.

In economic terms, one can explain the signs of derivative (13) noting that higher values of δ anticipate the repayment of a payg-ndc scheme's debt, while lower values postpone such repayment. The scheme gains by anticipating when the debt is more expensive than the wage bill growth rate, or looses when the debt is cheaper.

The signs of derivative (13) provide policy indications. Where social and political pressure would impose a $\pi > \omega$ rate, one could limit the resulting deficit by selecting 'high' values for δ.[7] On the other hand, where a $\pi < \omega$ rate should be chosen in order to produce surpluses designed to obtain, or increase, a buffer fund, one could expedite the process by setting δ at 'low' values.

3 Admitting variable longevity

We now want to investigate the effects of abandoning the assumption of constant in favor of monotonically variable longevity. We shall analyze two cases. The first assumes pefectly imputed longevity. In other words, it assumes that payg-ndc scheme is endowed with 'perfect foresight' (infallible forecasting tools) and accordingly can assign appropriate forward-looking conversion coefficients to each cohort that are based on survival rates the cohort itself will experiment in the future. The second case assumes lagged imputation of longevity, i.e. that the scheme assigns backward-looking

[7] Such values might easily match social consensus since they higher replacement rates by frontloading the pension profile.

coefficients to each cohort relaying on survival rates experimented by previous cohorts in the past.

3.1 Perfect imputation

Perfectly imputed longevity ensures actuarial fairness (benefits that are equivalent to contributions plus interest) but not, unfortunately, sustainability of the scheme. In fact, the following proposition holds

Proposition 4: in a $n = 1$ type payg-ndc scheme rewarding contributions at a $\pi = \omega$ rate,

$$\hat{\mathbf{s}}_t = \mathbf{s}_t > \mathbf{s}_{t\text{-}1}, \forall t \;\Rightarrow\; B_t > 0\,, \forall t$$
$$\hat{\mathbf{s}}_t = \mathbf{s}_t < \mathbf{s}_{t\text{-}1}, \forall t \;\Rightarrow\; B_t < 0\,, \forall t$$

(perfectly imputed longevity generates surpluses if longevity increases or deficits if it decreases).

Proof. In case $m = 2$, equation (11) holds that may now be converted into

$$(14) \qquad B_t = a\,W_t \left\{ 1 - \left[\frac{(1+\delta)(1+\delta+\hat{s}_{t-1,3}) + s_{t-1,3}(1+\delta+\hat{s}_{t,3})}{(1+\delta+\hat{s}_{t,3})(1+\delta+\hat{s}_{t-1,3})} \right] \right\} ,$$

which shows that B_t takes the opposite sign to that of the difference between the numerator and the denominator of the fraction (in square brackets) on the right side. Such a difference is as follows

$$(15) \qquad (1+\delta)\cdot(s_{t-1,3} - \hat{s}_{t,3}) + \hat{s}_{t,3}(s_{t-1,3} - \hat{s}_{t-1,3}).$$

According to present assumptions, $s_{t-1,3} = \hat{s}_{t-1,3}$ and $\hat{s}_{t,3} = s_{t,3}$. Therefore, difference (15) reduces to

$$(1+\delta)\cdot(s_{t-1,3} - s_{t,3}) \;,$$

which is negative when longevity increases and positive when it decreases. Extending the proof to the case when $m > 2$ only entails formal complications. **QED**.

One can intuitively explain Proposition 4 as follows: forward looking coefficients adjust (decrease or increase) benefits of each new cohort before the cohort itself can complete a different (longer or shorter) life span than the previous one, hence before a change (rise or fall) in the overall number of retirees has the time to occur.

3.2 Lagged imputation

Perfectly imputed longevity, assumed by Proposition 4, is a pure abstraction. One should admit forecasting errors raising problems of social and political acceptability. That is why European payg-ndc schemes[8] have chosen backward-looking coefficients. In fact, such schemes impute cohort retiring at time *t* the following table of survival probabilities

[8] European countries that have adopted ndc pension plans are Italy, Sweden, Poland and Latvia. For a comparison between Sweden and Italy, see Gronchi-Nisticò (2006).

(16) $\quad \hat{\mathbf{s}}_t = \overline{\mathbf{s}}_\tau$,

whose element related to age *a* is as follows

$$\overline{s}_{\tau,a} = s_{\tau,n+1\to n+2} \cdot s_{\tau,n+2\to n+3} \cdot \ldots \cdot s_{\tau,a-1\to a} \ ,$$

where $s_{\tau,j\to j+1}$ denotes ex-post survival rate according to which individuals alive in year $\tau \le t$ survived from age *j* to age $j+1$. Note that, under the working hypotheses assumed in § 1, the first rate refers to the cohort who retired in year $\tau-1$, the second in year $\tau-2$ and so on. The last rate refers to cohort who retired in year $\tau-\left[a-\left(n+1\right)\right]$.

Assuming survival probabilities (16) is equivalent to imputing longevity to cohorts with a lag.

Backward looking coefficients based on lagged tables (16) violate actuarial fairness since they generate a final debt (4) that is negative in case longevity increases or positive in case it is decreases. Nor such coefficients can guarantee sustainability. In fact, one can prove the following

Proposition 5: in a $n=1$ type payg-ndc scheme rewarding contributions at a $\pi=\omega$ rate,

$$\hat{\mathbf{s}}_t = \overline{\mathbf{s}}_\tau \ , \mathbf{s}_\mathrm{t} > \mathbf{s}_\mathrm{t\text{-}1} \ , \forall t \ \Rightarrow \ B_t < 0 \ , \forall t$$
$$\hat{\mathbf{s}}_t = \overline{\mathbf{s}}_\tau \ , \mathbf{s}_t < \mathbf{s}_{t\text{-}1} \ , \forall t \ \Rightarrow \ B_t > 0 \ , \forall t$$

(lagged imputation causes deficits if longevity increases or surpluses if it decreases).

Proof. Without loss of generality, one can assume $\tau = t$ (survival rates are collected in real time) such that, in case $m=2$, lagged imputation hypothesis implies

$$\hat{s}_{t,3} = s_{t-1,3}$$
$$\hat{s}_{t-1,3} = s_{t-2,3} \ .$$

Substituting into difference (15) yields

(17) $\quad s_{t-1,3}\left(s_{t-1,3} - s_{t-2,3}\right)$.

While proving Proposition 4 we showed that B_t takes the opposite sign to that of difference (17). We then proved present proposition since difference (17) is positive if longevity increases and negative if it decreases. Extending the proof to the case where $m>2$ only involves formal complications. **QED**.

3.3 Limiting disequilibria

Proposition 4 and 5 announce that both forward and backward looking conversion coefficients produce disequilibria. We want to prove that all of them can be limited by selecting high values for δ, i.e. by frontloading pension profiles.

Let us first consider unbalances announced by Proposition 4 for the case when longevity is perfectly imputed. We want to prove the following

Proposition 6: in a $n=1$ type payg-ndc scheme rewarding contributions at a $\pi=\omega$ rate,

$$\hat{\mathbf{s}}_t = \mathbf{s}_t \text{ , } \mathbf{s}_t > \mathbf{s}_{t-1} \;\Rightarrow\; \frac{\partial B_t}{\partial \delta} < 0$$

$$\hat{\mathbf{s}}_t = \mathbf{s}_t \text{ , } \mathbf{s}_t < \mathbf{s}_{t-1} \;\Rightarrow\; \frac{\partial B_t}{\partial \delta} > 0$$

(positive balance, i.e. surplus, produced by increasing longevity is a decreasing function of δ, while negative balance, i.e. deficit, produced by decreasing longevity is an increasing function of δ).

Proof. In case $m = 2$ the balance takes the form (11), whose derivative with respect to δ is as follows

$$\text{(18)} \qquad \frac{\partial B_t}{\partial \delta} = a\,W \cdot \left[\frac{s_{t-1,3}}{\left(1+\delta+\hat{s}_{t-1,3}\right)^2} - \frac{\hat{s}_{t,3}}{\left(1+\delta+\hat{s}_{t,3}\right)^2} \right].$$

Perfectly imputed longevity implies $\hat{s}_{t-1,3} = s_{t-1,3}$ and $\hat{s}_{t,3} = s_{t,3}$. Hence

$$\text{(19)} \qquad \frac{\partial B_t}{\partial \delta} = a\,W \left[\frac{s_{t-1,3}}{\left(1+\delta+s_{t-1,3}\right)^2} - \frac{s_{t,3}}{\left(1+\delta+s_{t,3}\right)^2} \right].$$

One can prove that

$$\text{(20)} \qquad \frac{\partial \dfrac{s_3}{\left(1+\delta+s_3\right)^2}}{\partial s_3} > 0 \,,$$

where s_3 assumes values $s_{t,3}$ and $s_{t-1,3}$ included in equation (19). In fact, such a derivative is

$$\text{(21)} \qquad \frac{\left(1+\delta+s_3\right)^2 - 2s_3\left(1+\delta+s_3\right)}{\left(1+\delta+s_3\right)^4}.$$

Since δ is included in field (7) while

$$\text{(22)} \qquad s_3 \in [0,1] \,,$$

as it is convenient for a survival rate, $1+\delta+s_3 > 0$ such that derivative (21) reduces to

$$\text{(23)} \qquad \frac{1+\delta-s_3}{\left(1+\delta+s_3\right)^3} \,,$$

which is positive since fields (7) and (22) implies $s_3 < 1+\delta$.

Let us now distinguish two cases: (i) if longevity increases, according to (20),

$$\text{(24)} \qquad \frac{s_{t,3}}{\left(1+\delta+s_{t,3}\right)^2} > \frac{s_{t-1,3}}{\left(1+\delta+s_{t-1,3}\right)^2} \,,$$

such that derivative (19) is negative; (ii) if longevity decreases, still according to (20), inequality (24) is reversed, such that derivative (19) is positive. **QED**.

Let us now consider unbalances announced by Proposition 5 for the case when longevity is imputed with a lag. We want to prove the following

Proposition *7*: in a $n = 1$ type payg-ndc scheme, rewarding contributions at a $\pi = \omega$ rate,

$$\hat{\mathbf{s}}_t = \bar{\mathbf{s}}_t \text{ , } \mathbf{s}_t > \mathbf{s}_{t-1} \;\Rightarrow\; \frac{\partial B_t}{\partial \delta} > 0$$

$$\hat{\mathbf{s}}_t = \vec{\mathbf{s}}_t \text{ , } \mathbf{s}_t < \mathbf{s}_{t-1} \;\Rightarrow\; \frac{\partial B_t}{\partial \delta} < 0$$

(negative balance, i.e. deficit, produced by increasing longevity is an increasing function of δ while positive balance, i.e. surplus, produced by decreasing longevity is a decreasing function of δ).

Proof. Lagged imputation of longevity implies $\hat{s}_{t-1,3} = s_{t-2,3}$ and $\hat{s}_{t,3} = s_{t-1,3}$. Therefore derivative (18) becomes

$$\text{(25)} \qquad \frac{\partial B_t}{\partial \delta} = a\,W \cdot \left[\frac{s_{t-1,3}}{\left(1+\delta+s_{t-2,3}\right)^2} - \frac{s_{t-1,3}}{\left(1+\delta+s_{t-1,3}\right)^2} \right].$$

which is positive if longevity increases, in which case $s_{t-1,3} > s_{t-2,3}$, or negative if it decreases, in which case $s_{t-1,3} < s_{t-2,3}$. **QED**.

4 Concluding remarks

Under increasing longevity, payg-ndc scheme can assign to each cohort both backward and forward conversion coefficients. The former reflect 'observed' longevity of previous cohorts. The latter reflect 'forecast' longevity of the cohort itself.

Backward looking coefficients prevent the scheme from being both actuarially fair and sustainable (even though it rewards contributions according to wage bill growth rate). The latter (aside from forecasting errors) can ensure fairness, not sustainability as well. In fact, they produce surpluses.

Policy maker faces two alternatives. One is to abandon the idea of a self-sufficient payg-ndc scheme and allow for general tax revenue pay structural deficits engendered by backward-looking coefficients.

Alternatively, one can opt for fairness that forward-looking coefficients aim to assure. However, a criterion would be necessary for equitably share resulting surpluses.

One can limit both deficits and surpluses by frontloading pension profiles, i.e. by selecting high values for δ parameter included in coefficients formula.

In 2003, five years after the ndc reform, Sweden implemented a third choice. While retaining the backward-looking conversion coefficients, it adopted the 'balance mechanism', an actuarial device designed to counter deficits by empirically adjusting the rate of return to long-run budget perspectives. [9]

[9] See Settergren-Mikula (2006) and Gronchi-Nisticò (2006). The balance mechanism also aims to counter unbalances procured by 'behavioral changes' (such as changing in wage curves and retirement age) already mentioned in footnote 1.

Bibliography

Settergren O. and B. D. Mikula (2006), "The Rate of Return of Pay As You Go Pension Systems", in R. Holzmann and E. Palmer, *Pension Reform: Issues and Prospects for Non-Financial Defined Contribution (NDC) Scheme,* Washington D.C.: World Bank,pp.117-142.

Gronchi S. and S. Nisticò (2006), "Implementing the NDC Theoretical Model: A Comparison of Italy and Sweden" in R. Holzmann and E. Palmer, *Pension Reform: Issues and Prospects for Non-Financial Defined Contribution (NDC) Scheme,* Washington D.C.: World Bank, pp.493-515.

Gronchi S. and S. Nisticò (2008), "Theoretical foundations of pay-as-you-go defined-contribution pension schemes", in *Metroeconomica*, Vol. 59, n. 2, pp. 131-159.

Valdés Prieto S. (2000), "The Financial Stability of Notional Account Pensions", *The Scandinavian Journal of Economics*, Vol. 102, n. 3, pp. 395-417.